Homework 11, Math 5032, Due March 14th

In the following, we repeat some of the arguments made in class for another functor. We will assume that A is a commutative ring with 1. If M is any module, let $F_\bullet$ as usual denote a free resolution of M:

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0.$$

You do not have to write all the gory details, just enough to convince Rob that you have understood what to do.

1. Let N be another module. Show that $\mathrm{Hom}_A(F_\bullet, N)$ defined as,

$$0 \to \mathrm{Hom}(F_0, N) \to \mathrm{Hom}(F_1, N) \to \cdots \to \\ \to \mathrm{Hom}(F_{n-1}, N) \to \mathrm{Hom}(F_n, N) \to \cdots$$

is a complex.

2. Define $\mathrm{Ext}^n_A(M, N)$ to be

$$H_n(\mathrm{Hom}_A(F_\bullet, N)) = \frac{\mathrm{Ker}(\mathrm{Hom}(F_n, N) \to \mathrm{Hom}(F_{n+1}, N))}{\mathrm{Im}(\mathrm{Hom}(F_{n-1}, N) \to \mathrm{Hom}(F_n, N))}.$$

Show that $\mathrm{Ext}^n_A(M, N)$ is independent of the free resolution.

3. If $0 \to N_1 \to N_2 \to N_3 \to 0$ is exact, show that for any module M there is a long exact seqence,

$$0 \to \mathrm{Hom}(N_3, M) \to \mathrm{Hom}(N_2, M) \to \mathrm{Hom}(N_1, M) \to \mathrm{Ext}^1_A(N_3, M) \to \\ \cdots \to \mathrm{Ext}^n_A(N_2, M) \to \mathrm{Ext}^n_A(N_1, M) \to \mathrm{Ext}^{n+1}_A(N_3, M) \to \cdots$$

4. This is a collection of exercises on injective modules. A module I is called injective if given any inclusion $0 \to N \xrightarrow{i} M$ and a homomorphism $f : N \to I$ there exists a homomorphism $g : M \to I$ such that $g \circ i = f$.

   (a) Show that a module I is injective if and only if given any ideal $J \subset A$ and a homomorphism $f : J \to I$, there exists an element $x \in I$ such that $f(a) = ax$ for all $a \in J$. (Zorn's lemma)

   (b) Show that if $A = \mathbb{Z}$, a module is injective if and only if it is divisible. (Recall that a module I over $\mathbb{Z}$ is divisible if for any $0 \neq n \in \mathbb{Z}$, the multiplication map $n : I \to I$ is surjective.)

   (c) If M is a $\mathbb{Z}$-module, show that $M \subset I$ where I is injective. (Zorn's lemma)

   (d) If I is injective over $\mathbb{Z}$ and A is as above, show that $\mathrm{Hom}_{\mathbb{Z}}(A, I)$ has a natural structure of an A-module and it is injective over A. (Slightly hard).

   (e) Using the above show that if M is any A-module, $M \subset I$ for some injective module I over A.