Homework 12, Math 5032, Due March 21st

In the following, $k = \mathbb{C}$ and G a finite group of order n. Unless otherwise mentioned, all vector spaces will be finite dimensional over k.

1. Let V be a representation of G (as always, this means that we are given a homomorphism of groups $\rho : G \to \mathrm{Aut}_k(V)$). Show that $S^2(V), \wedge^2(V)$ are both naturally G-modules and show that

$$\chi_{S^2V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V(g^2)\right)$$
$$\chi{\wedge^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right)$$

2. Let G act on a finite set S and let V be the vector space with basis S. Then V is a G-module. Show that $\chi_V(g)$ is the cardinality of $\{s \in S \mid gs = s\}$.

3. Let V, W be two G-modules. Show that

$$\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g).$$

4. Let $G = S_3$ be the permutation group on three elements $S = \{s_1, s_2, s_3\}$. Then as we have seen the three dimensional vector space on S is a G-module. Let $\sigma, \tau \in G$ with $\sigma^2 = e, \tau^3 = e$ and $\sigma\tau\sigma = \tau^{-1}$ as usual.

   (a) Show that G has two non-isomorphic one dimensional represeantations, one the trivial representation and the other given by $f : G \to \mathbb{C}^*$, where $f(\tau) = 1, f(\sigma) = -1$.

   (b) Show that the 1-dimensional subspace of V generated by $\sum s_i$ is a G-submodule.

   (c) Show that the 2-dimensional G-module $W = V/\mathbb{C}(\sum s_i)$ is an irreducible represenataion of G. We will later see that these are all the irreducible representaions of G upto isomorphism.