Homework 12, Math 5032

In the following, $k = \mathbb{C}$ and G a finite group of order n. Unless otherwise mentioned, all vector spaces will be finite dimensional over k.

1. In the following, let $G = S_d$, the permutation group on d elements and let H be the alternating group on d elements. If $x \in H$, let C_x denote its conjugacy class in G and D_x its conjugacy class in H.

   (a) If $x \in H$ is written as product of disjoint cycles of length $b_1, \ldots, b_k$ with $\sum b_i = d$, and if any b_i is even or $b_i = b_j$ for some $i \neq j$, show that $C_x = D_x$.

   (b) With the notation as above, in the rest of the cases, show that $D_x \neq C_x$ and $C_x - D_x$ is another conjugacy class with cardinality equal to that of D_x.

   (c) Let $d = 5$. Then show that H has five conjugacy classes.

   (d) Let V be a representation of G. Then it is also a representation of H. Let L be the 1-dimensional non-trivial representation of G. Show that V and $V \otimes L$ are isomorphic as representaions of H.

   (e) Write down a character table for the irreducible representations of H using the irreducible representations of G. (The dimensions are $1, 4, 5, 3, 3$.)

2. Let $R = k[x_1, x_2, x_3, y_1, y_2, y_3]$ be the polynomial ring in six variables and let I be the ideal generated by the three 2×2 minors of the matrix,

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}$$

   Show that kernel of the natural surjective homomorphism $R^3 \to I$ given by the basis in R^3 going to the three minors is a free module of rank 2.

3. Let R be any ring and let $S = R[x]$, polynomial ring in one variable. We consider R as an S-module via $R = S/xS$. Calculate $\mathrm{Tor}_i^S(R, R)$ for all i.

4. Let $R = k[\![x, y]\!]/(f(x, y))$ and let $k = R/(x, y)$. Try to calculate $\mathrm{Tor}_i^R(k, k)$ for all i.