**Homework 4, Math 5032, Due Feb 18th**

1. If $K \subset L$ is an extension of finite fields, show that the norm map $N_{L/K} : L^* \to K^*$ is onto. Deduce Hilbert 90, even though $K$ may not contain the required roots of unity.

2. Using the fact that an odd prime $p$ is a sum of squares if and only if it is $p \equiv 1 \bmod 4$, prove that the cokernel of the norm map $N : \mathbb{Q}(i)^* \to \mathbb{Q}^*$ is an infinite dimensional vector space over $\mathbb{F}_2$.

3. An exact sequence from Galois cohomology. If $G$ is a group (usually written multiplicatively) and $A$ is an abelian group (usually written additively), we say that $A$ is a $G$-module to mean that we are given an action of $G$ on $A$, which is same as saying that $A$ is a module over $\mathbb{Z}[G]$, the group ring. A homomorphism $\phi : A \to B$ between two $G$-modules is a $G$-module homomorphism if $\phi(ga) = g\phi(a)$ for all $g \in G$ and $a \in A$.

   (a) If $\phi : A \to B$ is a $G$-module homomorphism, show that there are natural induced homomorphisms (of abelian groups) $\phi^* : H^i(G, A) \to H^i(G, B)$ for $i = 0, 1$.

   (b) Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be an exact sequence of $G$-modules. Show that there exists a boundary homomorphism of abelian groups $\partial : H^0(G, C) \to H^1(G, A)$ giving a long exact sequence,

$$0 \to H^0(G, A) \xrightarrow{i^*} H^0(G, B) \xrightarrow{\pi^*} H^0(G, C) \xrightarrow{\partial} H^1(G, A) \xrightarrow{i^*} \\ H^1(G, B) \xrightarrow{\pi^*} H^1(G, C)$$

4. Galois Theory for inseparable extensions: If $L$ is a field, recall that a derivation $D$ of $L$ is an additive map $D : L \to L$ with $D(1) = 0$, satisfying the Leibniz' rule, namely $D(ab) = aD(b) + bD(a)$ for all $a, b \in L$. If $K \subset L$ is a field extension, we say that a derivation $D$ is a $K$-derivation, if $D$ is $K$-linear. Let $\mathfrak{D}_K(L)$ denote the set of all $K$-derivations of $L$, which is naturally an $L$-vector space.

   (a) Let $L$ be a field of positive charactersitc $p$. If $D$ is a derivation of $L$, show that $D(a^p) = 0$ for all $a \in L$.

   (b) Let $\mathfrak{D}_K(L)$ be as above. Show that if $D_1, D_2 \in \mathfrak{D}_K(L)$, so is the bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. (So that $\mathfrak{D}_K(L)$ is a Lie Algebra over $K$). Show that $D_1^p \in \mathfrak{D}_K(L)$. This property is called $p$-closedness. Thus $\mathfrak{D}_K(L)$ is a $p$-closed Lie Algebra.

   (c) Now assume that $L^p \subset K$ (in view of the first exercise, derivations can not detect such elements anyway) where $K \subset L$ is a finite extension. Let $L = K(a_1, \ldots a_n)$, where $n$ is minimal. Show that $\dim_L \mathfrak{D}_K(L) = n$ and $[L : K] = p^n$.

(d) Conversely, let $\mathfrak{D}$ be a finite dimensional (over L) vector space of $p$-closed Lie algebra of derivations of $L$. Show that if we define $K = \{a \in L \mid D(a) = 0 \forall D \in \mathfrak{D}_K(L)\}$ then $K \subset L$ is a subfield and it is a finite extension, $L^p \subset K$ and $\mathfrak{D} = \mathfrak{D}_K(L)$.

5. We have seen in class that if $K$ is a field and $n$ a positive integer which is not divisible by the characteristic of $K$ and $\omega$ is a primitive $n^{\text{th}}$ root of 1, then $K(\omega)$ is an abelian extension of $K$ with Galois group a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$.

   (a) In the above assume that $n = p$ is a prime. Show that the Galois group is cyclic.

   (b) Let $K$ be any field, $p$ any prime and $a \in K$. Show that if the polynomial $X^p - a$ is not irreducible over $K$ then it has a root in $K$. (Hint: If $p =$ the charactersitc of $K$, this is trivial. If not, use the previous part.)