Homework 5, Math 5032, Due Feb 25th

All rings considered will be commutative with 1 unless otherwise mentioned.

1. Let G be a finite group acting on a ring A (as ring automorphisms). Let $A^G \subset A$ be the subring of invariants.
   (a) Show that A is integral over A^G.
   (b) Show that if A is an integrally closed domain, so is A^G.
   (c) Deduce that $A = \mathbb{C}[x, y, z]/(z^n - xy)$ is an integrally closed domain. (Hint: $A \subset \mathbb{C}[u, v] = B$ with a suitable action of a cyclic group of order n on B and A is the ring of invariants.)

2. Decide which of the following rings are integrally closed.
   (a) $\mathbb{Z}[i]$
   (b) $\mathbb{Z}[\sqrt{2}]$
   (c) $\mathbb{Z}[\sqrt{13}]$
   (d) $\mathbb{C}[x, y]/(x^2 - y^3)$

3. If $K \subset A$ is a subring of $K[t]$, a polynomial ring in one variable t over a field K and $A \neq K$, show that $K[t]$ is integral over A.

4. Let K be a ring and let V be a finitely generated module over K. Let A be the ring of K-module endomorphisms of V (this is a non-commutative ring in general). One has the natural ring homomorphism $K \to A$, where $a \mapsto$ the scalar multiplication by $a \in K$. Show that any element $\alpha \in A$ is integral over K.