Homework 6, Math 5032, Due March 3rd

All rings considered will be commutative with 1 unless otherwise mentioned.

1. If A is an integrally closed domain, show that $A[x]$, the polynomial ring in the variable x is integrally closed.

2. Let K be finite extension of $\mathbb{Q}$ and let A be the integral closure of $\mathbb{Z}$ in K. A is usually called the ring of (algebraic) integers in K.

   (a) Show that A is a free abelian group of rank $[K : \mathbb{Q}]$.

   (b) Show that A_P, the localization of A at a maximal ideal P is a dvr.

   (c) If $K = \mathbb{Q}(\omega)$ where ω is a primitive third root of 1, show that $A = \mathbb{Z}[\omega]$.

   (d) Let $\sigma_i, i = 1, \ldots, n$ be the distinct embeddings of K in $\mathbb{C}$. Under the embedding $A \xrightarrow{\phi} \mathbb{C}^n$, given by

   $$a \mapsto (\sigma_1(a), \ldots, \sigma_n(a)),$$

   show that the image is a lattice. That is, $\phi(A)$ is a free abelian subgroup of $\mathbb{C}^n$ and that given any bounded set $B \subset \mathbb{C}^n$, there are only finitely many $a \in A$ such that its image under the above map is in B.

   (e) Let U be the group of units in A. Consider the map $U \to \mathbb{R}^n$ by

   $$u \mapsto (\log|\sigma_1 u|, \ldots, \log|\sigma_n u|).$$

   Show that this is a map of abelian groups, the image is a lattice in $\mathbb{R}^n$ and the kernel of this map is finite. Deduce that U is a finitely generated abelian group.

   (f) If $A = \mathbb{Z}[\omega]$, with ω a primitive third root of 1 as before, show that the group of units in A is a cyclic group of order 6.