Homework 7, Math 5032, Due March 17th

1. Let A be an affine algebra over a field k and $\overline{k}$ be an algebraic closure of k. Show that an element $f \in A$ is nilpotent if and only if for any k-algebra homomorphism $\phi : A \to \overline{k}$, $\phi(f) = 0$.

2. Let A be a Noetherian domain such that for any prime ideal $P \neq 0$, A_P is a dvr. (In particular all non-zero prime ideals are maximal). Such rings are called Dedekind domains. Let K be its fraction field.

   (a) If $0 \neq I \subset A$ is an ideal, show that there are only finitely prime ideals P_i such that $I \subset P_i$. Further show that there exists well defined integers $n_i > 0$ such that $I = \prod P_i^{n_i}$.

   (b) If $0 \neq I \subset K$ is a finitely generated A-submodule, show that the set $I^{-1} = \{x \in K \mid xI \subset A\}$ is an$A$-module. Show that I^{-1} is finitely generated by first showing that if $I \subset J$, then $J^{-1} \subset I^{-1}$ and if $0 \neq a \in K$, then $(aA)^{-1} = a^{-1}A$.

   (c) Show that if I is as above, $(I^{-1})^{-1} = I$.

   (d) Let $\operatorname{Pic} A$ denote the set of isomorphism classes of non-zero finitely generated A-submodules of K. That is, the class of two such are the same if and only if they are isomorphic as A-modules. If I is such a submodule, we denote by $[I]$ its class in $\operatorname{Pic} A$. If we define a multiplication on $\operatorname{Pic} A$ by $[I][J] = [IJ]$, show that it is well defined and makes $\operatorname{Pic} A$ into an abelian group with identity element $[A]$ and $[I]^{-1} = [I^{-1}]$. This is called the Picard group or Class group.

   (e) Show that $\operatorname{Pic} A$ is the trivial group if and only if A is a UFD (and hence a pid).

3. Let $A = \mathbb{C}[x, y]/(f)$ where $f = (y^2 - x^3 + 1)$.

   (a) Show that A is a Dedekind domain.

   (b) Let T be the set of all $\mathbb{C}$-derivations of A into A. Recall that these are the set of $\mathbb{C}$-vectorspace endomorphisms of A satisfying the Leibniz rule and T is naturally an A-module. Show that any such derivation can be written as $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ where $a, b \in \mathbb{C}[x, y]$ and $D(f) \equiv 0 \bmod f$.

   (c) Show that there is a map $E : \mathbb{C}[x, y] \to A$ of the form $a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ with $a, b \in \mathbb{C}[x, y]$ such that $E(f) = 1 \in Af$. Deduce that T is a free module of rank one over A.