Homework 8, Math 5032, Due March 24th

1. Let $R = (R, +, \cdot)$ be a ring, where $+, \cdot$ represent the addition and multiplication of R. Show that $R^{\mathrm{op}} = (R, +, \circ)$ where $a \circ b$ is defined to be $b \cdot a$ for all $a, b \in R$ is ring. If M is a left module over R, show that M is naturally a right module over R^{op} via the action $ma = am$ for all $a \in R, m \in M$.

2. Let R be a ring (with 1) and let M be a (left) R-module.

   (a) Define $\mathrm{Ann}\, M = \{a \in R \mid aM = 0\}$. Show that $\mathrm{Ann}\, M$ is a two-sided ideal. One says that a left module M is faithful if $\mathrm{Ann}\, M = 0$.

   (b) Assume M as above is faithful and simple. (A ring with such a module is called left-primitive). Let $D = \mathrm{End}_R M$, a division ring by Schur's lemma. Show that R is a division ring if $\dim_D M = 1$.

   (c) Ler R be as above. If for all $a, b \in R$, $a(ab - ba) = (ab - ba)a$, show that R is a division ring.

   (d) Again, R as above. Asuume that $1 + a^2$ is a unit for any $a \in R$. Show that R is a division ring.

3. In a ring R, an element of the form $ab - ba$ for $a, b \in R$ is called an additive commutator and if they are units, the element $aba^{-1}b^{-1}$ is called a multiplicative commutator. The latter is often just called a commutator and denoted $[a, b]$. Show that an element in a division ring is in the center if and only if it commutes with all additive commutators.

4. Let D be a division ring and assume that $a, c \in D$ such that $ac \neq ca$. Let $b = a - 1$. Then $b \in D^*$.

   (a) Show that $a([a^{-1}, c] - [b^{-1}, c]) = 1 - [b^{-1}, c] \neq 0$.

   (b) Using the above, show that an element c is in the center if and only if it commutes with all commutators.