Homework 9, Math 5032, Due March 31st

A will denote a commutative ring with 1 unless otherwise mentioned.

1. Let K be a field and V, W finite dimensional vector spaces over K.

   (a) If V, W have basis $\{v_i\}, \{w_j\}$, show that $V \otimes_K W$ is a vector space with basis $\{v_i \otimes w_j\}$.

   (b) Show that $\operatorname{Hom}(V, W)$ is naturally isomorphic to $V^\vee \otimes W$ where $V^\vee$ denote the dual vector space.

   (c) Let$\{v_i^\vee\}$ be the dual basis of $V^\vee$. Under the identification of $V^\vee \otimes V$ with $\operatorname{End} V$ identify the identity element in this ring as an element of the tensor product with respect to the basis $\{v_j^\vee \otimes v_i\}$.

2. (a) Compute $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

   (b) Show that if $0 \to M \to N \to P \to 0$ is any exact sequence of $\mathbb{Z}$-modules, the induced complex

   $$0 \to M \otimes_{\mathbb{Z}} \mathbb{Q} \to N \otimes_{\mathbb{Z}} \mathbb{Q} \to P \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

   is exact.

3. Let $\mathcal{Q}$ be the Quaternion algebra over $\mathbb{R}$.

   (a) Show that $\mathcal{Q} \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $M_2(\mathbb{C})$, the ring of 2×2 matrices over $\mathbb{C}$ as rings.

   (b) Show that $\mathcal{Q} \otimes_{\mathbb{R}} \mathcal{Q}^{\text{op}}$ is isomorphic to $M_4(\mathbb{R})$ as rings.

   (c) More generally let D be a division ring with center K. Assume that $\dim_K D = n < \infty$. Show that $D \otimes_K D^{\text{op}}$ is isomorphic as ring to $M_n(K)$.