Algebraic Geometry, Math 539, Homework 1

1. Let X be a smooth projective surface and $C \subset X$ a smooth curve. Using the exact sequence (which you should know),

$$0 \to \mathcal{O}_C(-C) \to \Omega^1_X \mid C \to \Omega^1_C \to 0,$$

show that $K_C = \Omega^1_C = K_X \otimes \mathcal{O}_C(C)$, where as usual $K_X = \wedge^2 \Omega^1_X$, the canonical line bundle of X.

2. Let C be a smooth projective curve and let $\Delta \subset C \times C$ be the diagonal. Using the above, compute Δ^2, the intersection number in terms of the genus of C.

3. Let X be a smooth projective surface and let $C, D \subset X$ be two curves. For a point $P \in X$, assume that P is isolated in $C \cap D$. Define $I(P, C, D) = \ell(\mathcal{O}_{X,P}/\mathcal{O}(-C) + \mathcal{O}(-D))$, the local intersection multiplicity where as usual ℓ denotes the length. If $C \cap D$ is a finite set of points, show that $(C \cdot D) = \sum_{P \in C \cap D} I(P, C, D)$.

4. Let notation be as above. Show that $I(P, C, D) = 1$ if and only if $P \in C \cap D$ and both C, D are smooth at P and the local equations of C, D at P form a local co-ordinate system for X.

5. Let $C, D \subset \mathbb{P}^2$ be two curves with no common components with $\deg C = d, \deg D = e$. Show that $(D \cdot C) = de$. This is known as Bezout's theorem, which started all of interesection theory.

6. Let $C, D \subset \mathbb{P}^2$ be smooth curves of degrees d, e respectively. Assume that $C \cap D$ has cardinality de and assume that they are defined by polynomials F, G. If $H = 0$ defines a curve E passing through all the points of $C \cap D$, show that $H = AF + BG$ for suitable polynomials A, B. (This is a slightly weak version of what is known as Max Noether's Theorem).

7. Using the above, deduce the following: If $C, D \subset \mathbb{P}^2$ are two smooth cubics meeting in nine distinct points and if E is a third cubic passing through eight of these nine points, then it also passes through the ninth.