Algebraic Geometry, Math 539, Homework 2

1. Let X be a smooth projective curve and K its canonical line bundle. Show that $K + D$ is free for any divisor with $\deg D \geq 2$ and it is very ample for $\deg D \geq 3$. (These are the first and trivial cases of Fujita and Mukai conjectures).

2. (Halphen's Theorem). Show that if X is as above and $P_i \in X, 1 \leq i \leq g+3$ where g is the genus of X and P_i's are 'general' then $D = \sum P_i$ is very ample and thus every smooth curve can be embedded in $\mathbb{P}^3$.

3. For this problem, assume that the base field k is arbitrary (not necessarily algebraically closed). Let X be a smooth projective curve over this field and assume that the genus is zero. Then show that $-K$ is very ample and gives an emddeing of X in $\mathbb{P}^2$ as a conic. Show that X is isomorphic to $\mathbb{P}^1$ if and only if X has a rational point over k. That is, there exists a morphism $\operatorname{Spec} k \to X$ over k.

4. I assume all of you know the definition (and properties of) multiplicity $m_P(C)$ of a curve C at a point P. If C is an irreducible curve, define $m(C) = \max\{m_P(C) \mid P \in C\}$ (Why is this finite?). If X is a (projective) scheme and L is a line bundle on X, show that L is ample if and only if there exists an $\epsilon > 0$ such that for all irreducible curves $C \subset X$, $(L \cdot C) \geq \epsilon m(C)$. (This is called Seshadri's criterion for ampleness).