Homework 9

(1) If α is algebraic over a field K with $\deg \alpha$ defined as the degree of $\operatorname{Irr}(\alpha, K, X)$ odd, show that $K(\alpha) = K(\alpha^2)$.

(2) Let K be a field and $\overline{K}$ its algebraic closure. Show that if K is finite, $\overline{K}$ is countable and if K is infinite, cardinality of $K, \overline{K}$ are the same.

(3) Let K be a finite field (necessarily of characteristic p, a prime).
  (a) Show that cardinality of K is $q = p^n$ for some n.
  (b) Show that the map $\operatorname{Frob} : K \to K$ $\operatorname{Frob}(x) = x^p$ is an automorphism and thus every element in K has a unique p^{th} root. (The above map is called the *Frobenius*.)
  (c) Show that $\operatorname{Frob}(x) = x$ if and only if $x \in \mathbb{F}_p \subset K$.

(4) Let G be a finite group and E a field on which G acts as field automorphisms, faithfully. (*Faithful* means if for a $\sigma \in G$, $\sigma(x) = x$ for all $x \in E$, then $\sigma = e$.) Let $K = E^G = \{x \in E \mid \sigma(x) = x, \text{for all } \sigma \in G\}$.
  (a) Show that E is algebraic over K.
  (b) Show that there exists an $x \in E$ such that $\sigma(x) \neq x$ for all $e \neq \sigma \in G$.

(5) Let K be a field and $L = K(t)$ rational functions in t.
  (a) If $\alpha \in L$ is algebraic over K, show that $\alpha \in K$.
  (b) If $K \subset E \subset L$ is a subfield with $K \neq E$, show that t is algebraic over E.
  (c) If E as above is $K(\beta)$ where $\beta \notin K$ is of the form $f(t)/g(t)$, f, g coprime, show that $[L : E] = \max\{\deg f, \deg g\}$. (An important theorem called Lüroth's theorem says E is always of the above form.)

(6) Let $A \subset B$ be two commutative rings. An A-module map $d : B \to B$ is called an A-*derivation* (written a derivation if A is understood) if it satisfies the Leibniz rule: $d(ab) = ad(b) + bd(a)$ for all $a, b \in B$.
  (a) Show that $d(a) = 0$ for all $a \in A$.
  (b) Let $\operatorname{Der}_A(B) = D$ be the set of all A-derivations of B. Show that it is naturally a B-module. (The additive structure on D and action of B on it should be obvious and natural.)
  (c) If $d, e \in D$, show that the Lie bracket $[d, e]$ defined as $[d, e](a) = d \circ e(a) - e \circ d(a)$ is also in D.
  (d) If characteristic of A is a prime $p > 0$, and $d \in D$, show that d^p (composing d with itself p times) is also a in D.

(e) If A, B are fields and the extension is separable, show that $D = 0$.

(f) If $A = K(t), B = K(u)$ with K a field of characteristic $p > 0$ and the inclusion is given by $t \mapsto u^p$, show that there is an A-derivation d on B with $d(u) = 1$.