Nagata's Theorem

Nagata's Theorem:

Let R be an integral domain and $p \in R$ be a prime element. Let $S = \{1, p, p^2, \ldots\}$ and let $T = S^{-1}R$. If T is a UFD then so is R.

Proof. We will identify R as a subring of T since the natural map $R \to T$ is injective.

First we show that given a prime element $q' \in T$, there exists a prime element $q \in R$ such that its image in T is q' up to a unit in T. Given such a q', we can multiply it by p^n for large enough n and get an element $q \in R$ such that $q = p^n q'$ in T. Further, we may assume that p does not divide q in R. If we show that q is a prime in R, we would be done. So, assume q divides ab in R, then q divides a or b in T, say a. Write $a = sq$ for some $s \in T$. Then again, we have $c = p^n s \in R$ for large n and thus we get $p^n a = cq$ in R. But, p is a prime and does not divide q implies p^n divides c in R. So writing $c = p^n d$ with $d \in R$, we see that $a = dq$, proving that q is a prime.

Now, let $0 \neq x \in R$ and write x as a product of primes up to unit in T. From the previous part, thus we can write $x = uq_1q_2 \cdots q_m$ where $q_i \in R$ are primes different from p and u is a unit in T. Again, we can find an n so that $c = p^n u \in R$ and thus we get $p^n x = cq_1 \cdots q_m$ in R. As before, we see that p^n divides c and thus we get an equation $x = dq_1 \cdots q_m$. Notice that d is a unit in T. So, we have $de = 1$ for some $e \in T$ and again, for suitable n we get $e' = p^n e \in R$. So, we have $de' = p^n$ and then it is clear that, since p is a prime, $d = p^r$ up to a unit and so $x = p^r q_1 \cdots q_m$ up to a unit and we are done. $\square$