Examples of monoids

(1) $\mathbb{N} = \{0, 1, 2, \ldots\}$ is a monoid with respect to addition. Similarly, $\mathbb{N}_+ = \mathbb{N} - \{0\}$ and $\mathbb{N}$ are both monoids with respect to multiplication.
(2) For any set S, $\operatorname{End} S$, the set of all maps from S to itself, called endomorphisms, is a monoid with respect to composition.
(3) The same is true in many situations with extra structure. For example, the set of all vector space endomorphisms from a vector space V to itself, the set of all continuous functions from a topological space to itself, the set of all polynomial maps from say $\mathbb{C}$ to itself etc., are all monoids with respect to composition.

Examples of groups

(1) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all abelian groups with respect to the usual addition.
(2) If K is any field (for example, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), then K^*, the set of non-zero elements in K is an abelian group with respect to multiplication.
(3) If S is any set, the set of all bijections from S to itself is a group with respect to composition operation.
(4) If X is any topological space, the set of all homeomorphisms from X to itself is a group with respect to composition.
(5) If K is any field, the set $M_{m\times n}(K)$ of all $m \times n$ matrices over K is a an abelian group with respect to matrix addition.
(6) Again, if K is a field, $Gl_n(K)$, the set of all $n \times n$ matrices over K with non-zero determinant is a group with respect to multiplication.

    Similarly, $Sl_n(K)$, square matrices with determinant one, D, diagonal matrices with non-zero determinant, $U_n(K)$, upper triangular matrices with non-zero determinant etc.
(7) The unit circle, which we think of all complex numbers z with $|z| = 1$, is an abelian group with respect to the usual multiplication of complex numbers.

Examples of rings

(1) $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all commutative rings with respect to the usual addition and multiplication.
(2) If $n > 1$ is an integer, the set $\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \ldots, n-1\}$ is a commutative ring with respect to operations modulo n. That is, if we take $a, b \in \mathbb{Z}/n\mathbb{Z}$, we can find a unique element $c \in \mathbb{Z}/n\mathbb{Z}$

such that $a + b \equiv c \bmod n$. So define $a + b = c$. Similarly, we can find a unique element $d \in \mathbb{Z}/n\mathbb{Z}$ such that $ab \equiv d \bmod n$. Define $ab = d$.

(3) The set of all functions from a set S to any commutative ring R is a commutative ring, if we define for two such functions f, g, $(f + g)(s) = f(s) + g(s)$ and $(fg)(s) = f(s)g(s)$.

(4) $M_n(K)$ for any field is a (non-commutative, if $n > 1$) ring, with respect to the usual matrix addition and multiplication.

(5) If $z \in \mathbb{C}$, then

$$\mathbb{Z}[z] = \{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \in \mathbb{C} | a_i \in \mathbb{Z}\},$$

for all possible $n \geq 0$ is a commutative ring with respect to the addition and multiplication in $\mathbb{C}$.

The same could be done if you replace $\mathbb{Z}$ with $\mathbb{Q}$ or $\mathbb{R}$. Of course, if we take $\mathbb{C}$ instead, we just get all of $\mathbb{C}$.

(6) For any commutative ring, denote by $R[[x]]$ the set of all functions from $\mathbb{N}$ to R. Then we can define addition by $(f+g)(n) = f(n) + g(n)$ and multiplication by $(fg)(n) = \sum_{i=0}^{n} f(i)g(n-i)$. This makes $R[[x]]$ into a commutative ring, called the ring of *formal power series.*

An analogous ring is the ring of germs of all analytic (resp. holomorphic) functions at a point in $\mathbb{R}$ or $\mathbb{C}$ (resp. $\mathbb{C}$).

(7) Letting R to be as above, now consider $R[x]$ to be the set of all functions $f : \mathbb{N} \to R$ such that $f(n) = 0$ for all large n. Then the above operations make $R[x]$ into a commutative ring, called the *polynomial ring.*

(8) More generally, let M be any monoid (operation written multiplicatively) and consider $R[M]$ to be the set of all maps $f : M \to R$ such that $f(m) = 0$ for all but finitely many elements $m \in M$. Then we can endow it with a ring structure as follows. For $f, g \in R[M]$, define $(f + g)(m) = f(m) + g(m)$ and $(fg)(m) = \sum_{p,q,pq=m} f(p)g(q)$. In the special case when M is a group, we call $R[M]$ to be the *group ring.*

In the above cases, it is typical to write an element f as $f = \sum_{n=0}^{\infty} f(n)x^n$, for polynomial ring and power series ring and for $R[M]$, $f = \sum_{m \in M} f(m)m$. With this representation, addition and multiplication look more natural.

(9) Consider $\mathbb{C}[x]$, polynomial ring in one variable, x. Then the set of all $\mathbb{C}$-linear maps from $\mathbb{C}[x]$ to itself is a ring with respect to the obvious addition and composition as multiplication. Then, $\mathbb{C}[x]$ is a *subring* of this ring in a natural way, since multiplication by polynomials is a $\mathbb{C}$-linear map. We can consider a

bigger ring containing $D = d/dx$, which is also a $\mathbb{C}$-linear map. You can see that $Dx - xD = 1$ in this ring, in particular, it is non-commutative. (Sometimes, this ring is called *Weyl algebra*).

Examples of fields

(1) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.
(2) For any prime number p, $\mathbb{Z}/p\mathbb{Z}$ is a field. We usually denote this by $\mathbb{F}_p$, a finite field of p elements.
(3) For a complex number z, $\mathbb{Q}[z]$ is a field if and only if z is *algebraic*. That is, z satisfies an algebraic equation of the form $z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$ where $a_i \in \mathbb{Q}$. For example, $i = \sqrt{-1}$ is algebraic and so is $\sqrt{2}$.