Final, Math 5031, Due Dec 14th, 4pm

(1) (a) Prove that any group of order p^2q, p, q primes has a non-trivial normal subgroup.
   (b) Describe all groups of order 55.

(2) (a) Let R be a commutative ring containing $\mathbb{C}$ as a subring. This makes R into a $\mathbb{C}$ vector space and further assume that $\dim_{\mathbb{C}} R < \infty$. Prove that R is Noetherian.
   (b) If $\mathfrak{m} \subset R$ is any maximal ideal, show that the natural localization map $j : R \to R_{\mathfrak{m}}$ is onto. (Hint: $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for large n and thus kernel of j contains $\mathfrak{m}^n$ for large n.)
   (c) Let R be as above and assume that R is reduced. That is, it has no non-zero nilpotent elements. Show that R is isomorphic to the product of finitely many copies of $\mathbb{C}$ as rings. (Hint: $\mathbb{C}$ is algebraically closed and Chinese remainder theorem.)

(3) (a) Let K be a field and let $f(T) \in K[T]$ be a polynomial of degree n and let E be the splitting field (contained in the algebraic closure of K) of f. Show that $[E : K] \leq n!$.
   (b) Let S_n, the symmetric group on n letters act on $E = \mathbb{C}(x_1, \ldots, x_n)$ by permuting the variables and let $K = E^{S_n}$, the fixed field. Show that $f(T) = \prod_{i=1}^{n}(T - x_i) \in K[T]$. Show that for $1 \leq k \leq n$, $s_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} \in K$.
   (c) Show that E is the splitting field of $f(T)$ over K.
   (d) Show that $K = \mathbb{C}(s_1, s_2, \ldots, s_n)$.