Homework 1

(1) Let G, H be groups such that $(o(G), o(H))$ are relatively prime. Describe all group homomorphisms from G to H.

(2) Let $\mathbb{F}_q$ be a finite field with q elements. Calculate the order of the group $GL_n(\mathbb{F}_q)$. Find a p-Sylow subgroup of $GL_2(\mathbb{F}_p)$.

(3) Let G, H be groups (not necessarily finite) and let $\phi : G \to \mathrm{Aut}(H)$ be a group homomorphism, where as usual, $\mathrm{Aut}(H)$ is the group of all group automorphisms of H. Define a product on the set $H \times G$ (caution: we are just looking at it as a set) as follows.

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1\phi(g_1)(h_2), g_1g_2).$$

Show that this makes $H \times G$ into a group. (These are called *semi-direct products* and becomes identical to the direct product if $\phi(g) = Id$ for all $g \in G$.) Show that the subset $H \times e_G \subset H \times G$ is a normal subgroup and the quotient is naturally isomorphic to G.

(4) If a finite group acts on a set with n elements transitively, show that $o(G) \geq n$.

(5) If G is a p-group (for a prime number p as always), show that the group homomorphism $C : G \to \mathrm{Aut}(G)$, defined in class, via conjugation, is not injective (unless of course $G = \{e\}$).

(6) Show that if a group G has order p^2q where p, q are primes, either G has a non-trivial (which means, not G or $\{e\}$) normal subgroup or $o(G) = 12$. (The last case too has a non-trivial normal subgroup, which we will do in class, being a little tricky).

(7) Let H be a proper subgroup of a finite group G. Show that the union of all conjugates of H can not be G.