Homework 2

(1) If Aut(G) for a finite group is cyclic, prove that G is abelian.
(2) If $G =< a >$ is a finite cyclic group of order n, show that for any m, the map $f_m : G \to G$, given by $f_m(a) = a^m$ determines a group homomorphism. Calculate the order of Aut(G).
(3) Let G be a group and H a subgroup of G of finite index. Show that there exists a normal subgroup of G contained in H and also of finite index.
(4) Do problems 40, 41 (page 79) from the book.
(5) Here I discuss another important group from classical groups, the special unitary group $SU(2)$. These are all 2×2 complex matrices A such that $\overline{A}^t A = Id$ with $\det A = 1$, where the bar indicates complex conjugation and the upper t indicates transpose. Thus, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$\overline{A} = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}, \overline{A}^t = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

(a) Show that the map $SU(2) \to \mathbb{C}^2$, given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b)$$

identifies $SU(2)$ with the real 3-sphere S^3. (Thus we get a surprising group structure on S^3 and as some of you may know, this is very rare.)

(b) For any $A \in SU(2)$, describe the conjugacy class containing it in terms of S^3. This should be a 'nice' subset of S^3. You may assume (not a very difficult fact) that any such A is conjugate to a diagonal matrix in $SU(2)$.