Homework 3

(1) We define an abelian group D to be *divisible* if for any non-zero integer n, multiplication by n on D is surjective. In other words, given any $a \in D$, there exists $b \in D$ such that $nb = a$. For example, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both divisible and so is $\mathbb{C}^*$, non-zero complex numbers with respect to multiplication.

Let D be a divisible group and let A be a subgroup of an abelian group B. Given any homomorphism $f : A \to D$, show that there exists a homomorphism $g : B \to D$ such that $g(a) = f(a)$ for all $a \in A$. (If you are unfamiliar with transfinite induction, i. e. Zorn's lemma, you may assume B and thus A are finitely generated abelian groups.) Deduce that if D is a subgroup of an abelian group G, then G has a subgroup H such that the natural map $D \oplus H \to G$ is an isomorphism.

(2) Let $I(p)$ for a prime p denote the set of all elements a in $\mathbb{Q}/\mathbb{Z}$ such that $p^n a = 0$ for some n. Show that $I(p)$ is a divisible subgroup and $\mathbb{Q}/\mathbb{Z} = \oplus_p I(p)$.

(3) Show that the natural map

$$\mathrm{Hom}(\oplus_{i=1}^n G_i, \oplus_{j=1}^m H_j) \to \oplus_{i,j}\mathrm{Hom}(G_i, H_j)$$

is an isomorphism of groups, where G_i, H_j are abelian groups.

(4) Show that for any positive integer n, there exists a unique cyclic subgroup of order n in $\mathbb{C}^*$.

(5) For a finite abelian group G define its dual $G^\vee$ to be the group $\mathrm{Hom}(G, \mathbb{C}^*)$. Show that $o(G) = o(G^\vee)$.

(6) Show that the natural map $G \to G^{\vee\vee}$ given below is an isomorphism. For any $g \in G$, we get an element of $G^{\vee\vee}$ as follows. Define a map $G^\vee \to \mathbb{C}^*$ by for $\phi \in G^\vee$ maps to $\phi(g)$.