Homework 4

(1) Let G be a group. We define a category $\mathcal{S}_G$ as follows. The objects are pairs (S, σ) where S is a set and $\sigma : G \to \text{Aut}(S)$ is a group homomorphism. That is, we have an action of G on S. If $(S, \sigma), (T, \tau)$ are two such, define $\text{Mor}((S, \sigma), (T, \tau))$ as the set of all set maps $f : S \to T$ such that $f(gs) = gf(s)$ for all $g \in G, s \in S$. Show that these make $\mathcal{S}_G$ into a category.

(2) Notation as above and if H is another group and $\phi : G \to H$ is a group homomorphism, define $F : \mathcal{S}_H \to \mathcal{S}_G$ as follows. If (S, σ) is an object in $\mathcal{S}_H$, define $F((S, \sigma)) = (S, \sigma \circ \phi)$. Similarly, if $f : (S, \sigma) \to (T, \tau)$ is a morphism in $\mathcal{S}_H$, define $F(f) : F((S, \sigma)) \to F((T, \tau))$ by the map $f : S \to T$. Show that this does define a functor.

(3) Let $f : R \to S$ be a homomorphism of commutative rings. Show that, if $P \subset S$ is a prime ideal, so is $f^{-1}(P) = \{a \in R | f(a) \in P\}$. Give an example where $f^{-1}(M)$ may not be maximal, even if M is.

(4) If R is a commutative ring (with 1), an element $a \in R$ is called *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$. Let N (or $N(R)$) be the set of all nilpotent elements in R.
  (a) Show that N is an ideal.
  (b) Show that N is contained in every prime ideal. (We will show the converse when we do localizations.)

(5) Let R be as above. An element $a \in R$ is called a *unit* if there exists a $b \in R$ such that $ab = 1$. Let $U(R)$ denote the set of all units in R.
  (a) Show that $U(R)$ is an abelian group with respect to multiplication.
  (b) If $R = \mathbb{Z}[i]$, the ring of Gaussian integers, which are the set of complex numbers $a + bi$, $a, b \in \mathbb{Z}$ and $i = \sqrt{-1}$, show that $U(R)$ is a cyclic group of order 4.
  (c) (Little harder). Let $R = \mathbb{Z}[\sqrt{2}]$, set of real numbers of the form $a + b\sqrt{2}$, with $a, b \in \mathbb{Z}$. Easy to see that this is a ring with respect to the usual operations. Show that $U(R)$ is an infinite (abelian) group. (Hint: What is the order of $1 + \sqrt{2} \in U(R)$?). (This is often referred to as solutions of Pell's equation, though Pell had nothing to do with it.)
  (d) Show that for any commutative ring, $U(R)$ contains a subgroup denoted by $1 + N = \{1 + x | x \in N(R)\}$.

(e) If $f(x) = a_0 + a_1x + \cdots + a_nx^n \in U(R[x])$ is a unit in a polynomial ring in x, show that $a_0 \in U(R)$ and $a_i \in P$ P any prime ideal for all $i > 0$. Conversely, if $a_0 \in U(R)$ and $a_i, i > 0$ are nilpotent, show that $f(x) \in U(R[x])$. (As I said, once we do localization, these conditions are the same).