Homework 5

(1) Let $S_i, i = 1, 2$ be the circles of radius i with center the origin. Find a polynomial $P(x, y)$ or prove its existence with the property that $P(a, b) = a$ for all $(a, b) \in S_1$ and $P(a, b) = b$ for all $(a, b) \in S_2$.

(2) Let R be any commutative ring which contains $\mathbb{F}_p$, the finite field with p elements as a subring. Show that the map $F : R \to R$ given by $F(x) = x^p$ for any $x \in R$ is a ring homomorphism. (This map is called the *Frobenius*). Show that for any maximal ideal $M \subset R$, $F^{-1}(M)$ is a maximal ideal.

(3) We just check some of the boring, but useful details about localization. So, R is a commutative ring, $S \subset R$ a multiplicatively closed subset and let $T = S^{-1}R$ with $j : R \to T$ the homomorphism discussed in class.

  (a) Show that $j(\alpha)$ is a unit in T for any $\alpha \in S$.

  (b) For any $t \in T$, show that there exists $a \in R, \alpha \in S$ such that $j(a) = j(\alpha)t$.

  (c) Let $\mathcal{I}(T)$ (resp. $\mathcal{I}(R)$) be the set of all proper ideals in T (resp. R). We have a natural map $j^* : \mathcal{I}(T) \to \mathcal{I}(R)$ given by $j^*(I) = j^{-1}(I)$. Show that this map is injective. Show that the image of this map is precisely the set of all ideals J of R such that $J \cap S = \emptyset$.

  (d) Show that the nil ideal $N \subset R$, the set of all nilpotent elements of R is precisely the intersection of all prime ideals of R. (Hint: If $a \in R$ is not nilpotent, consider the multiplicatively closed subset $\{1, a, a^2, \ldots\}$.)

(4) We next discuss an important, very simple rings, called *Discrete valuation rings*, dvr for short. Let K be any field and let $v : K^* = K - \{0\} \to \mathbb{Z}$ be any group homomorphism (which we will assume is non-trivial). Define $v(0) = +\infty$. We say such a v is a discrete valuation, if $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$.

  (a) Let $R = \{a \in K | v(a) \geq 0\}$. Show that R is a subring of K, called a dvr and fraction field of R is K.

  (b) Show that $a \in R$ is a unit if and only if $v(a) = 0$.

  (c) Show that R is a pid and it is a local domain with only two prime ideals, 0 and the maximal ideal.

  (d) If R is a subring of S, which in turn is a subring of K, show that $R = S$ or $S = K$. That is R is a maximal subring of K.

(e) Fix a prime number p. We can write any non-zero rational number r uniquely as $p^n a/b$ with $a, b \in \mathbb{Z}, b \neq 0$ and p does not divide a, b. Define $v(r) = n$ and show that it is a discrete valuation on $\mathbb{Q}$.

(f) Let K be any field and let $K((x))$ denote all Laurent series of the form $\sum_{n \in \mathbb{Z}} a_n x^n$, with $a_n \in K$ and $a_n = 0$ for all sufficiently small n. That is, $a_n = 0$ if $n < N$ (the N can vary). Show that $K((x))$ is a field with the usual addition and multiplication. For $0 \neq f(x) = \sum a_n x^n \in K((x))$, define $v(f(x))$ to be the smallest n such that $a_n \neq 0$. Show that v is a discrete valuation of $K((x))$ and the corresponding dvr is $K[[x]]$, the formal power series contained in $K((x))$.

(g) Let K be any field and $K(x)$ be the field of rational functions. If $r(x) = f(x)/g(x)$ with $f, g \in K[x]$, define $\deg r = \deg f - \deg g$. Show that the map given by $v(r) = -\deg r$ is a discrete valuation on $K(x)$.