Homework 6

(1) Let R be a commutative ring and S a multiplicatively closed set in R and without loss of generality, we will assume that $1 \in S$. As usual, we will denote for any R-module M, the natural map from $M \to S^{-1}M$ by j. So, $j(m) = [(m, 1)]$.
   (a) Show that the kernel of $j : M \to S^{-1}M$ is the set of all elements $m \in M$ such that $sm = 0$ for some $s \in S$.
   (b) Show that if $0 \to M \xrightarrow{i} N \xrightarrow{\pi} P \to 0$ is an exact sequence of R-modules, $0 \to S^{-1}M \xrightarrow{i'} S^{-1}N \xrightarrow{\pi'} S^{-1}P \to 0$ is an exact sequence of $S^{-1}R$-modules.
   (c) Show that if an R-module M is Noetherian, then $S^{-1}M$ is a Noetherian $S^{-1}R$-module.
   (d) Show that if M is an R-module, then $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset R$.
   (e) If M, N are R-modules, show that there is a natural homomorphism of $S^{-1}R$-modules, $S^{-1}\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$. Can you think of some conditions which will make this map an isomorphism?

(2) Recall that for a topological space X and a subset A, A is said to be *discrete* if for any point $a \in A$, there exists an open neighbourhood U of a such that $U \cap A = \{a\}$. Show that if $A \subset \mathbb{R}^n$ is a discrete subgroup, then A is a free abelian group. (Hint: Discrete subsets of compact sets are finite.)

(3) If R is a Noetherian ring and M a module over R, show that M is Noetherian if and only if M is finitely generated.

(4) Let R be a commutative ring and I an ideal. For a module M, as usual we denote by IM, the submodule of M generated by elements of the form am, where $a \in I, m \in M$. If M is finitely generated (very important) and $IM = M$, show that there exists an element $f \in R$ such that $1 - f \in I$ and $fM = 0$. (This is one version of Nakayama's lemma).

(5) Let R be a Noetherian ring and let $f : R \to S$ be a ring homomorphism. Then, we can view S as an R-module naturally and assume that S is a finitely generated R-module. Then show that for any $s \in S$, there exists a *monic* polynomial $P(X) \in R[X]$ such that $P(s) = 0$.

(6) Let the assumptions be as in the previous problem. Show that S is Noetherian.