<u>Homework 7</u>

(1) Let R be a principal ideal domain.
  (a) Let $\mathbf{a} = (a_1, \ldots, a_n)$ be a (non-zero) row vector with $a_i \in R$. Show that there exists a $\sigma \in GL_n(R)$ such that $\mathbf{a}\sigma = (d, 0, \ldots, 0)$, where of course d is the gcd of the a_is.
  (b) Next let N be a $k \times n$ matrix with $k \leq n$ over R. Show that there exists $\sigma \in GL_n(R), \tau \in GL_k(r)$ such that the matrix $N' = \tau N \sigma$ has the form $N' = (A, 0)$ where A is a $k \times k$ matrix, 0 is the zero matrix of size $k \times n - k$ and A is a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_k$ with the property that d_i divides d_{i+1} for all i, with the understanding that if one $d_i = 0$, then so are all $d_j, j > i$.
  (c) Deduce (easily) the structure theorem for pid that we proved in class, just using the fact that submodule of a free module of rank n is free of rank at most n.

(2) Let R be an Euclidean domain. Recall that this means there is a function $d : R - \{0\} \to \mathbb{N}$ such that given $a \neq 0, b \in R$, there exists $q, r \in R$ such that $b = qa + r$ with either $r = 0$ or $d(r) < d(a)$. You may also assume that $d(ab) \leq d(a)d(b)$. We fix an $n > 0$ and denote by $E_{ij}(a)$ with $i \neq j, a \in R$ to be the $n \times n$ matrix which has 1 s on the diagonal, a at the ij th place and zero elsewhere. Notice that $\det E_{ij}(a) = 1$. Show that $SL_n(R)$ is generated by $E_{ij}(a)$ for all possible $i \neq i, a \in R$.

(3) Let R be a commutative integral domain. We say that R is a *Dedekind domain* if all the non-zero prime ideals are maximal and for any such prime P, R_P, the localization is a discrete valuation ring. We also denote by K the fraction field of R.
  (a) If I is any non-zero ideal, denote by $I^{-1} = \{\alpha \in K | \alpha I \subset R\}$. Show that I^{-1} is an R-submodule of K and $II^{-1} = R$. Deduce that I (and I^{-1}) is finitely generated and thus R is Noetherian.
  (b) Show that for any set of distinct non-zero prime ideal $P_1, \ldots, P_n$, $\cap P_i = \prod P_i$.
  (c) Show that any non-zero ideal is contained in at most finitely many prime ideals.
  (d) Show that any non-zero ideal I can be uniquely written as $I = P_1^{r_1} \cdots P_n^{r_n}$, where P_i s are maximal ideals and $r_i \in \mathbb{N}$.