Homework 8

(1) Let A be an integral domain. Describe all ring automorphisms $\phi : A[X] \to A[X]$ and $\psi : A[[x]] \to A[[x]]$ with $\phi(a) = a, \psi(a) = a$ for all $a \in A$.

(2) Let M be a module over $R = k[[x_1, \ldots, x_n]]$, k a field, which is complete with respect to the maximal ideal $\mathfrak{m}$ generated by the x_i s. Assume that $M/\mathfrak{m}M$ is a finite dimensional vector space over $R/\mathfrak{m} = k$. Show that M is finitely generated over R. (One can easily deduce Weierstrass preparation theorem for power series rings over a field from this.)

(3) Let R be an UFD and K its fraction field. Show that if $x \in K$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$ with $a_i \in R$, then $x \in R$. (This is usually described as UFDs are *integrally closed*).

(4) Let $R = \mathbb{C}[x, y], S = \mathbb{C}[[x, y]]$ and $F = y^2 - x^3, G = xy + (x+y)^3$.
   (a) Show that F, G are irreducible in R.
   (b) Show that neither $R/FR, R/GR$ is a UFD.
   (c) Show that F is irreducible but G is not in S.