Midterm 1, Algebra 5031, Oct 8, 2015

Answer as many questions as you can.

(1) Let G be a group with trivial center. Let $\operatorname{Aut} G$ be the group of automorphisms of G as a group. That is, the set of all $f : G \to G$, group automorphisms, with respect to composition. Show that the center of $\operatorname{Aut} G$ is also trivial. (Hint: Conjugation automorphisms)

(2) Let G be a finite group and let p be the smallest prime dividing the order of G. If G has a subgroup H of index p, show that H is normal in G. (Hint: Action on conjugates of H)

(3) If F is a free abelian group on a finite set $S \subset F$ and $v = \sum a_s \cdot s \in F$ where $a_s \in \mathbb{Z}, s \in S$, show that $F/\mathbb{Z} \cdot v$ is free if and only if $\gcd(a_s)_{s \in S} = 1$. (Hint: Torsion free=free)

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a *continuous* group homomorphism. Show that $f(x) = xf(1)$ for all $x \in \mathbb{R}$. (Hint: $f(r) = rf(1)$ for $r \in \mathbb{Q}$)

(5) Describe a left ideal of $M_2(K)$, the ring of 2×2 matrices over a field K which is neither zero nor all of $M_2(K)$. (Hint: Zero columns)

(6) Let $R = \mathbb{Z}/10\mathbb{Z}$ and let $S = \{1, 5, 5^2, \ldots\} \subset R$. How many elements does $S^{-1}R$ have? (Hint: j surjective)

(7) (a) State the division algorithm for $R[x]$, polynomial ring in one variable over a commutative ring R.
   (b) Show that $K[x]$ is a principal ideal domain, where K is a field.