Please return your answers to me on the first day of class. If you are unfamiliar with any of the concepts, that is fine. This is just to let me know what you are comfortable with.

(1) Let S be any finite set with n elements and let $\mathrm{Aut}S$ denote the set of all *bijections* from S to itself. Calculate the cardinality of $\mathrm{Aut}S$.

(2) Let V be a finite dimensional vector space over (say) $\mathbb{R}$. If $f : V \to V$ is a vector space homomorphism, show that f is injective if and only if it is surjective.

(3) Calculate the product of the polynomials $x - 1$ and $x^2 + 1$.

(4) Find all 2×2 matrices M over the reals such that M^2 is the identity matrix.

(5) Calculate the determinant of the matrix M,

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$$

If possible, find a 3×3 matrix N such that MN is the identity matrix.

(6) Let p be a fixed prime number and define a function on all non-zero rational numbers $v_p : \mathbb{Q} - \{0\} \to \mathbb{Z}$ as follows. If $a \in \mathbb{Z}$ non-zero, there exists a unique non-negative integer $v_p(a)$ such that $p^{v_p(a)}$ divides a, but $p^{v_p(a)+1}$ does not. If $r \in \mathbb{Q} - \{0\}$, we can write $r = a/b$ with $a, b \in \mathbb{Z}$, both non-zero. So, define $v_p(r) = v_p(a) - v_p(b)$. For convenience, define $v_p(0) = +\infty$. Show that the *distance* defined by $|x - y|_p = p^{-v_p(x-y)}$ for any $x, y \in \mathbb{Q}$ makes $\mathbb{Q}$ into a metric space. Is $\mathbb{Q}$ complete with respect to this metric?

(7) Denote by R, the set of all functions from $\mathbb{N} = \{0, 1, 2, \ldots\}$, the set of all non-negative integers, to the field of complex numbers $\mathbb{C}$. Show that the operations $f + g, fg$ for elements $f, g \in R$ defined as $(f+g)(n) = f(n)+g(n)$, $(fg)(n) = \sum_{i=0}^{n} f(i)g(n-i)$ for any $n \in \mathbb{N}$ makes R into a ring. Show that $fg = gf$ for any f, g. Verify that the function $f(n) = 0$ for all n is the additive zero and $g(n) = 0$ for all $n > 0$ and $g(0) = 1$ is the multiplicative identity. Finally, show that if $h \in R$ with $h(0) \neq 0$, then it has a multiplicative inverse.