DEGENERATING FAMILIES OF RANK TWO BUNDLES

N. MOHAN KUMAR, CHRIS PETERSON, AND A. PRABHAKAR RAO

ABSTRACT. We construct families of rank two bundles $\mathcal{E}_t$ on $\mathbf{P}^4$, in characteristic two, where for $t \neq 0$, $\mathcal{E}_t$ is a sum of line bundles, and $\mathcal{E}_0$ is non split. We construct families of rank two bundles $\mathcal{E}_t$ on $\mathbf{P}^3$, in characteristic p, where for $t \neq 0$, $\mathcal{E}_t$ is a sum of line bundles, and $\mathcal{E}_0$ is non split.

1. A DEGENERATING FAMILY OF SPLIT BUNDLES ON $\mathbf{P}^4$ IN CHARACTERISTIC 2

We will construct a family of rank two bundles $\mathcal{E}_t$ on $\mathbf{P}^4$, in characteristic two, where for $t \neq 0$, $\mathcal{E}_t$ is a sum of line bundles, and $\mathcal{E}_0$ is non split. The construction derives from Tango's example of a rank two bundle on $\mathbf{P}^5$ in characteristic two [5]. (For different constructions of such phenomena on $\mathbf{P}^4$ and $\mathbf{P}^3$, see [1], [2].)

Let $X = \mathbf{P}^4 \times \mathbf{A}^1$. Let $k_1, \dots, k_5$ be the degrees of forms a, b, c, d, e on $\mathbf{P}^4$ where the forms form a regular sequence. Impose the condition $k_1 = k_2 + k_5 = k_3 + k_4$. Let t be a parameter for $\mathbf{A}^1$. Consider the matrices

$$\Phi = \begin{bmatrix} 0 & t & -e & d \\ -t & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -t \\ c & e & t & 0 \end{bmatrix}.$$

These matrices give maps $\Phi : \mathcal{L}_1 \to \mathcal{F}$ and $\Psi : \mathcal{F} \to \mathcal{L}_2$, where

$$\mathcal{L}_1 = \mathcal{O}_X(k_2 - k_4) \oplus \mathcal{O}_X \oplus \mathcal{O}_X(-k_5) \oplus \mathcal{O}_X(-k_4) \tag{1.1}$$

$$\mathcal{F} = \mathcal{O}_X \oplus \mathcal{O}_X(k_2 - k_4) \oplus \mathcal{O}_X(k_3) \oplus \mathcal{O}_X(k_1 - k_5) \tag{1.2}$$

$$\mathcal{L}_2 = \mathcal{O}_X(k_2 - k_4 + k_1) \oplus \mathcal{O}_X(k_1) \oplus \mathcal{O}_X(k_2) \oplus \mathcal{O}_X(k_3). \tag{1.3}$$

Let Y be the hypersurface in X with equation $s = at - be + cd = 0$. Note that $\Phi\Psi = \Psi\Phi = sI$ and that both Ψ and Φ have rank exactly two at each point of Y. Consider the pull backs $\Phi^{(1)} : \mathcal{L}_1^{(1)} \to \mathcal{F}^{(1)}$ and $\Psi^{(1)} : \mathcal{F}^{(1)} \to \mathcal{L}_2^{(1)}$ of Φ and Ψ by the Frobenius morphism $F : X \to X$. Form the matrix

$$\Delta^{(1)} = \begin{bmatrix} s.I & \Phi^{(1)} \\ \Psi^{(1)} & s.I \end{bmatrix}.$$

Then $\Delta^{(1)}$ gives a map of bundles: $\mathcal{F}^{(1)}(-Y) \oplus \mathcal{L}_1^{(1)} \to \mathcal{F}^{(1)} \oplus \mathcal{L}_2^{(1)}(-Y)$.

Claim 1.1. The image $\mathcal{G}$ of $\Delta^{(1)}$ is a rank four sub-bundle of $\mathcal{F}^{(1)} \oplus \mathcal{L}_2^{(1)}(-Y)$.

Proof. At a point of X where $s \neq 0$, it is clear that $\Delta^{(1)}$ has rank at least four. At a point where $s = 0$, both $\Phi^{(1)}$ and $\Psi^{(1)}$ have rank two. Hence $\Delta^{(1)}$ has rank at least

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four at every point of X. But $\Delta^{(1)}\Delta^{(1)} = 0$, hence by the rank-nullity theorem, $\Delta^{(1)}$ has rank exactly equal to four at each point of X. □

Now impose the following additional condition on the degrees k_i:

$$k_1 = 2(k_4 - k_2).$$

With these conditions, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\vee : \mathcal{O}_X(-k_1) \to \mathcal{F}^{(1)}(-Y) \oplus \mathcal{L}_1^{(1)}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} : \mathcal{F}^{(1)} \oplus \mathcal{L}_2^{(1)}(-Y) \to \mathcal{O}_X.$$

It is immediate to see that the composite maps $\mathcal{O}_X(-k_1) \to \mathcal{G}$ and $\mathcal{G} \to \mathcal{O}_X$ are respectively a vector bundle injection and surjection and that together we get a monad of vector bundles

$$0 \to \mathcal{O}_X(-k_1) \to \mathcal{G} \to \mathcal{O}_X \to 0.$$

We have thus created a family of rank two bundles $\mathcal{E}_t$ on $\mathbf{P}^4$.

When $t \neq 0$, it is evident from the matrix $\Delta^{(1)}$ that the restriction $\mathcal{G}_t$ on $\mathbf{P}^4$ is the sum of four line bundles $\mathcal{G}_t = \mathcal{O}_{\mathbf{P}^4}(2k_3 - k_1) \oplus \mathcal{O}_{\mathbf{P}^4}(k_1 - 2k_5) \oplus \mathcal{O}_{\mathbf{P}^4}(-k_1) \oplus \mathcal{O}_{\mathbf{P}^4}$. Hence $\mathcal{E}_t$ is split on $\mathbf{P}^4$ as $\mathcal{E}_t = \mathcal{O}_{\mathbf{P}^4}(2k_3 - k_1) \oplus \mathcal{O}_{\mathbf{P}^4}(k_1 - 2k_5)$.

Claim 1.2. When $t = 0$, $\mathcal{E}_0$ is not split.

Proof. It is enough to show that $\mathcal{G}_0$ is not a sum of four line bundles on $\mathbf{P}^4$. For then, by Horrock's theorem, some twist of $\mathcal{G}_0$ will have a nontrivial intermediate cohomology group. Hence, using the monad which gives $\mathcal{E}_0$ from $\mathcal{G}_0$, it follows that $\mathcal{E}_0$ also has some non-zero intermediate cohomology in some twist, by which reason it is not split.

Now if $\mathcal{G}_0$ is the sum of four line bundles, since we are on $\mathbf{P}^4$, the inclusion $\mathcal{O}_{\mathbf{P}^4}(-k_1) \hookrightarrow \mathcal{G}_0$ of bundles must pick out a global section s which is part of a set of minimal generators for $H^0_*(\mathcal{G}_0)$. On the other hand, s is obtained as the sum of two sections given by the first and fifth columns of the matrix $\Delta^{(1)}_{t=0}$. Since each of these columns has only three non-zero polynomials (all non-constant), these two sections are not nowhere-vanishing. Hence neither section can be part of a set of minimal generators for $H^0_*(\mathcal{G}_0)$. Therefore their sum s cannot be one either, which is a contradiction. □

2. A degenerating family of split bundles on $\mathbf{P}^3$ in characteristic p

In characteristic p, let $q = p^r$ where $r \geq 1$ and $r \geq 2$ if $p = 2$. Choose positive integers k_1, k_2, k_3, k_4, k_5 satisfying

(2.1) $$k_3 = k_4$$

(2.2) $$k_1 = k_2 + k_5$$

(2.3) $$k_1 = k_3 + k_4$$

(2.4) $$k_1 = q(k_3 - k_2)$$

Solutions can always be found. In fact, choose integers $l, m > 0$ subject to the condition $(q-2)l = 2m$, and let $k_1 = ql, k_2 = m, k_3 = l+m, k_4 = l+m, k_5 = ql-m$. As in the previous section, choose a regular sequence of forms a, b, c, d, e on $\mathbf{P}^4$ with degrees k_1, k_2, k_3, k_4, k_5 and consider the matrices Φ, Ψ and the bundles $\mathcal{L}_1, \mathcal{F}, \mathcal{L}_2$

on $X = \mathbf{P}^4 \times \mathbf{A}^1$. Consider the pull backs $\Phi^{(r)} : \mathcal{L}_1^{(r)} \to \mathcal{F}^{(r)}$ and $\Psi^{(r)} : \mathcal{F}^{(r)} \to \mathcal{L}_2^{(r)}$ of Φ and Ψ by the Frobenius power $F^{(r)} : X \to X$. Form the matrix

$$\Delta^{(r)} = \begin{bmatrix} sI & \Phi^{(r)} \\ \Psi^{(r)} & s^{q-1}I \end{bmatrix}.$$

As before, $\Delta^{(r)}$ gives a map of bundles: $\mathcal{F}^{(r)}(-Y) \oplus \mathcal{L}_1^{(r)} \to \mathcal{F}^{(r)} \oplus \mathcal{L}_2^{(r)}(-Y)$ whose image is a rank four bundle $\mathcal{G}$. The numerical conditions allow maps

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\vee} : \mathcal{O}_X(-k_1) \to \mathcal{F}^{(r)}(-Y) \oplus \mathcal{L}_1^{(r)}$$

and

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} : \mathcal{F}^{(r)} \oplus \mathcal{L}_2^{(r)}(-Y) \to \mathcal{O}_X(qk_2).$$

It is immediately seen that the composite maps $\mathcal{O}_X(-k_1) \to \mathcal{G}$ and $\mathcal{G} \to \mathcal{O}_X(qk_2)$ are respectively a vector bundle injection and a vector bundle surjection on X.

Note that we do not have a monad on $\mathbf{P}^4 \times \mathbf{A}^1$. However, let us make a special choice for c, d such as $c = X_3^{k_3}$ and $d = X_4^{k_3}$ (where X_3 and X_4 are linear forms). If we now restrict to the hyperplane $X_3 = X_4$, which is $\mathbf{P}^3 \times \mathbf{A}^1$, we do get a monad. We thus get a family of rank two bundles, $\mathcal{E}_t$, on $\mathbf{P}^3 \times \mathbf{A}^1$.

All of the previous arguments carry through. When $t \neq 0$, $\mathcal{G}_t$ on $\mathbf{P}^4$ is a direct sum of line bundles, hence also its restriction to $\mathbf{P}^3$, and it is evident that the scalars in the matrices for the monad force $\mathcal{E}_t$ to be a split rank two bundle on $\mathbf{P}^3$. When $t = 0$, the rank four bundle, $\mathcal{G}_0$ on $\mathbf{P}^4$, is not a sum of line bundles as before. Hence neither is its restriction to $\mathbf{P}^3$. Therefore $\mathcal{E}_0$ is non split on $\mathbf{P}^3$.

3. A degenerating complete intersection curve

In this section, we will present in characteristic p a simple example of a family of complete intersection curves in $\mathbf{P}^3$ which degenerates to a subcanonical but non-complete intersection curve. This example is not one where the curves are smooth; we will present a smooth example in the next section. However, the simplicity of the algebra in this example indicates some phenomena which are harder to see in more complicated examples. Both of our examples (of this section and the next) are the end product of computer calculations using the degenerating sums of line bundles described in the preceding sections. In that sense, these examples are not informed by any geometric insight; rather they derive from extensive algebraic experimentation.

Let $\mathbf{P}^3$ have coordinates a, b, c, d, and let $s = a^{p-1}d^{p+1} - b^{p-1}c^{p+1}$. Consider the matrix

$$\psi = \begin{bmatrix} 0 & -s & -a^{(p-1)p^2} & -b^{(p-1)p^2} \\ s & 0 & -c^{(p+1)p^2} & -d^{(p+1)p^2} \\ a^{(p-1)p^2} & c^{(p+1)p^2} & 0 & -f \\ b^{(p-1)p^2} & d^{(p+1)p^2} & f & 0 \end{bmatrix},$$

where $f = (a^{(p-1)p^2}d^{(p+1)p^2} - b^{(p-1)p^2}c^{(p+1)p^2})/s$. Note that in the special case where the field has characteristic p, $f = s^{p^2-1}$.

It is immediate that the Pfaffian of this skew symmetric matrix is zero, and that the matrix has rank two at every point of $\mathbf{P}^3$. So the image of the map

$$\mathcal{O}_{\mathbf{P}^3}(2p^2) \oplus \mathcal{O}_{\mathbf{P}^3} \oplus 2\mathcal{O}_{\mathbf{P}^3}(2p-(p-1)p^2) \xrightarrow{\psi} \mathcal{O}_{\mathbf{P}^3}(2p) \oplus \mathcal{O}_{\mathbf{P}^3}(2p^2+2p) \oplus 2\mathcal{O}_{\mathbf{P}^3}(p^3+p^2),$$

is a rank two bundle $\mathcal{E}$ in any characteristic.

The columns of ψ give sections of $\mathcal{E}$, and the entries of a column give an ideal which defines the zero-scheme of the section scheme-theoretically. So the first column defines a subcanonical curve Y defined by the ideal $(s, a^{(p-1)p^2}, b^{(p-1)p^2})$. The degree of Y is easily computed: Since it is supported on $a = b = 0$, intersect it with the plane $c = 0$. On this plane, we may take $d = 1$, and s becomes a^{p-1} on this open set. So $Y \cap \{c = 0\}$ is the complete intersection of a^{p-1} and $b^{(p-1)p^2}$ on this open set. Therefore, Y has degree $(p-1)^2p^2$. A similar argument shows us that the curve obtained from the second column has degree $(p+1)^2p^2$. With this information about two different twists of $\mathcal{E}$, it is easy to find the Chern classes of $\mathcal{E}$ if they are desired.

Proposition 3.1. *Let Y_0 be the subcanonical curve in $\mathbf{P}^3$ described by the ideal $(s, c^{(p+1)p^2}, d^{(p+1)p^2})$. In characteristic p, Y_0 is the flat limit of a family of complete intersection curves.*

Proof. Consider the family of complete intersection curves given by the two forms

$$f_1 = d^{p(p+1)}t + b^{p(p-1)}s$$
$$f_2 = c^{p(p+1)}t + a^{p(p-1)}s.$$

This family of curves in $\mathbf{P}^3$ is defined over $\mathbf{A}^1_k - \{t = 0\}$, where k is the field of characteristic p. By the completeness of the Hilbert scheme, the family can be extended (uniquely) to a flat family of space curves over $\mathbf{A}^1_k$. We claim that the limit curve when $t = 0$ is Y_0.

Indeed let $\mathcal{Y}$ be the flat family over $\mathbf{A}^1_k$. The ideal of $\mathcal{Y}$, $I(\mathcal{Y})$ contains f_1, f_2. Also, since $\mathcal{Y}$ is flat over $\mathbf{A}^1_k$, multiplication by t is injective from $\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{Y}}$. Hence if $f \in k[a, b, c, d, t]$ and $tf \in I(\mathcal{Y})$, then $f \in I(\mathcal{Y})$. So we find elements in $I(\mathcal{Y})$ by 'saturating (f_1, f_2) with respect to t'. The following steps can be made (using algebra in characteristic p):

i) $f_1a^{(p-1)p} - f_2b^{(p-1)p} = s^pt$, hence $s^p \in I(\mathcal{Y})$.

ii) $f_1^p - b^{p^2(p-1)}s^p = d^{p^2(p+1)}t^p \in I(\mathcal{Y})$, hence $d^{p^2(p+1)} \in I(\mathcal{Y})$. Likewise, $c^{p^2(p+1)} \in I(\mathcal{Y})$.

iii) Hence trivially, $c^{p^2(p+1)}s, d^{p^2(p+1)}s \in I(\mathcal{Y})$.

Now the ideal of the limit curve certainly contains all elements obtained from taking elements in $I(\mathcal{Y})$ and setting $t = 0$. Hence in the ideal of the limit curve we find $a^{p(p-1)}s, b^{p(p-1)}s, c^{p^2(p+1)}s, d^{p^2(p+1)}s, c^{p^2(p+1)}, d^{p^2(p+1)}$. The first four tell that the ideal (saturated with respect to a, b, c, d) contains s. Hence the limit curve has ideal containing $s, c^{p^2(p+1)}, d^{p^2(p+1)}$. We conclude that the limit curve is a subscheme of the subcanonical curve Y_0.

Now the complete intersection has degree $p^2(p+1)^2$, hence also the flat limit curve. So does Y_0. The flat limit may have embedded components, so let C be the maximum locally Cohen-Macaulay subscheme of the flat limit. By degree reasons, $C = Y_0$, hence the flat limit is Y_0. □

Remark 3.2. We would like to make some general remarks here. The ideal of the complete intersection (f_1, f_2) is defined over $\mathbf{Z}$. It is plausible that because the iterated limit $lim_{t\to 0}(lim_{q\to p}\mathcal{Y}_{\mathbf{Z}})$ is subcanonical, so should be the iterated limit $lim_{q\to p}(lim_{t\to 0}\mathcal{Y}_{\mathbf{Z}})$, (where q represents an arbitrary prime). In the next section, where the limit curve Y_0 will be smooth as well, one may expect that $lim_{t\to 0}\mathcal{Y}_{\mathbf{Z}}$

should be smooth and subcanonical. Hence if it is not a complete intersection, we should expect an example in characteristic zero of a complete intersection family degenerating to a smooth non-complete intersection. We will show with some calculations that the flat family over $\mathbf{Z}$ does not specialize to the flat family over $\mathbf{F}_p$, hence such an expectation is unjustified.

Let us work with $p = 2$. Consider the ideal $(f_1 = d^6t + b^2s, f_2 = c^6t + a^2s)$ where $s = ad^3 - bc^3$, defined in $\mathbf{Z}[a, b, c, d, t]$. This gives a family of complete intersection curves defined over the open set U in $Z = Spec\ \mathbf{Z}[t]$ where $t \neq 0$. We will extend this to a t-flat family as follows:

Let $\bar{s} = ad^3 + bc^3$. Following the same elementary steps of saturating with respect to t, we find that the requirement that t is a non-zero divisor creates a larger ideal $(f_1, f_2, s\bar{s}, c^6\bar{s}, d^6\bar{s})$. If we specialize to $t = 0$, we get the ideal $(a^2s, b^2s, s\bar{s}, c^6\bar{s}, d^6\bar{s})$.

This last ideal is familiar since it is of the form $I_1s + I_2\bar{s}$, where $I_1 = (a^2, b^2, \bar{s})$ defines the same scheme as the ideal $(a^2, ab, b^2, \bar{s})$, giving a subcanonical curve obtained from a generalized null-corellation bundle, and likewise, $I_2 = (c^6, d^6, s)$. According to Schwartau's method of liaison addition [4], the ideal $I_1s + I_2\bar{s}$, defines a locally Cohen-Macaulay curve whose deficiency module is the direct sum of those of the curves from I_1, I_2, Now, we may easily compute the degree of this locally Cohen-Macaulay curve to be 36. Hence, this curve *is* the flat limit of our family as $t \to 0$. It is also well known that this curve is not subcanonical (there being no rank two bundles with such a deficiency module.) So in this case, we get a flat limit (in characteristic zero) which is not subcanonical nor a complete intersection.

At the same time, this curve with ideal $(a^2s, b^2s, s\bar{s}, c^6\bar{s}, d^6\bar{s})$ is defined for all primes other than 2. To extend flatly to the prime 2, we saturate with respect to 2. For example $d^6.b^2s + b^2.d^6\bar{s}$ yields $2d^6b^2ad^3$, hence we add the element ab^2d^9. A small calculation shows right away that the flat limit at $p = 2$ cannot be the curve Y_0 which is scheme theoretically generated by s, d^{12}, c^{12}. For consider the intersection of Y_0 with the plane $a = b$ The support is at $(1, 1, 0, 0)$ and is defined in variables c, d by the equations $c^3 + d^3 = 0, d^{12} = 0$, a complete intersection of degree 36. But if we toss ab^2d^9 into the ideal of Y_0, we add d^9 into these local equations reducing the degree to 27. Hence $ab^2d^9 \notin I(Y_0)$.

Hence we conclude that the flat family over $\mathbf{Z}$ does not specialize to the flat family over $\mathbf{F}_p$.

4. A smooth curve degeneration example

We will work out an example in characteristic 2. This example was created by studying the degenerating family of sums of line bundles in Section 1, where the forms are powers of linear forms of degrees $4, 1, 1, 3, 3$, and restricting to the hyperplane where the coefficient of t in s is zero. This results in the following situation.

Start with the rank two bundle $\mathcal{E}$ on $\mathbf{P}^3$ given by the image of the following matrix ψ

$$\begin{bmatrix} 0 & s & a^4 & b^4 \\ s & 0 & c^{12} & d^{12} \\ a^4 & c^{12} & 0 & s^3 \\ b^4 & d^{12} & s^3 & 0 \end{bmatrix} : \mathcal{O}_{\mathbf{P}^3}(8) \oplus 3\mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_{\mathbf{P}^3}(4) \oplus 3\mathcal{O}_{\mathbf{P}^3}(12),$$

where $s = ad^3 - bc^3$. $\mathcal{E}$ is generated by four global sections s_1, s_2, s_3, s_4 given by the columns of ψ, and by Kleiman's Bertini theorem [3], the general section of $\mathcal{E}(1)$ generated out of these four sections will have a smooth zero-scheme. It is also true from computer calculations that sections s_2, s_3, s_4 lift to the sections of a sum of line bundles, while specific combinations of s_1 lift to sections of the sum of line bundles. In fact, $a^2s_1, b^2s_1, c^6s_1, d^6s_1$ can all be lifted. In view of this, we define the following smooth curve and show that it is the flat limit of a complete intersection curve.

Theorem 4.1. *Let π be transcendental or sufficiently general over the field $\mathbf{F}_2$. Let $s = ad^3 + bc^3, l = c + \pi^4 d$. Then the curve Y in $\mathbf{P}^3$ with equations*

$$\begin{aligned} f_1 &= ls + a^4c + b^4d \\ f_2 &= (a^9 + b^9)s + c^{13} + d^{13} \\ f_3 &= (a^9 + b^9)a^4 + lc^{12} + ds^3 \\ f_4 &= (a^9 + b^9)b^4 + ld^{12} + cs^3 \end{aligned}$$

is subcanonical, smooth and though not a complete intersection, it is the flat limit of the family of complete intersection curves with equations

$$\begin{aligned} F_1 &= a^7t^3 + l(b^2s + d^6t) + c(a^4b^2 + a^2st + c^6t^2) + db^6 \\ F_2 &= b^7t^3 + l(a^2s + c^6t) + ca^6 + d(b^4a^2 + b^2st + d^6t^2) \end{aligned}.$$

Proof. It is evident that Y is the zero-scheme of the section $(a^9+b^9)s_1+ls_2+cs_3+ds_4$ of $\mathcal{E}(1)$, hence if it has the right codimension, Y is a subcanonical and non-complete intersection curve. We will leave to the end verification of the codimension, which will follow from the smoothness computation. It is easy to check that $\mathcal{E}(1)$ has second Chern class 49, hence Y will have degree 49.

We compute the flat extension $\mathcal{Y}$ of the family given by F_1, F_2 as usual. First rewrite F_1, F_2 as

$$\begin{aligned} F_1 &= b^2f_1 + (ld^6 + a^2cs)t + c^7t^2 + a^7t^3 \\ F_2 &= a^2f_1 + (lc^6 + b^2ds)t + d^7t^2 + b^7t^3 \end{aligned}.$$

i) $(a^2F_1 + b^2F_2)/t = (sf_1) + (a^2c^7 + b^2d^7)t + (a^9 + b^9)t^2 = G \in \mathcal{I}_\mathcal{Y}$;
ii) $(sF_1 + b^2G)/t = d^6f_1 + [c^7s + b^2(a^9 + b^9)]t + (a^7s)t^2 = G_2 \in \mathcal{I}_\mathcal{Y}$;
$(sF_2 + a^2G)/t = c^6f_1 + [d^7s + a^2(a^9 + b^9)]t + (b^7s)t^2 = G_3 \in \mathcal{I}_\mathcal{Y}$;
iii) $(d^6F_1 + b^2G_2)/t = f_4 + (c^7d^6 + a^7b^2s)t + a^7d^6t^2 \in \mathcal{I}_\mathcal{Y}$;
$(c^6F_2 + a^2G_3)/t = f_3 + (d^7c^6 + b^7a^2s)t + b^7c^6t^2 \in \mathcal{I}_\mathcal{Y}$;
iv) Lastly, $(c^6F_1 + d^6F_2 + sG)/t^2 = f_2 + (a^7c^6 + b^7d^6)t \in \mathcal{I}_\mathcal{Y}$;

Now setting $t = 0$, we find that the ideal of the flat limit contains each of the functions $a^2f_1, b^2f_1, c^6f_1, d^6f_1, f_2, f_3, f_4$. Hence the limit ideal, in its saturation, contains f_1, f_2, f_3, f_4. By degree reasons, the flat limit is the same as Y.

To establish smoothness of Y, we look at the Jacobian matrix with rows $\partial f_i, i = 1 \dots 4$.

First suppose we are at a point of Y where $l = 0$. The first row of the Jacobian reduces to $\begin{bmatrix} 0 & 0 & s + a^4 & * \end{bmatrix}$, while the third row is $\begin{bmatrix} a^{12} + d^4s^2 & * & * & * \end{bmatrix}$. Considering $f_4 + \pi^4 f_3$, we conclude that $(a^9 + b^9)(\pi a + b)^4 = 0$. If $(a^9 + b^9) = 0$, from f_2, we get $c^{13} + d^{13} = 0$, which together with $l = 0$ gives $c = d = 0$ (provided π does not satisfy the condition $1 + \pi^{52} = 0$). Since this forces both a and b to be non-zero, we see an obvious 2×2 minor of the Jacobian which does not vanish at this point.

On the other hand, if $a^9 + b^9 \neq 0$, it is easily seen that all four coordinates are non-zero and we have the point $(a, \pi a, \pi^4, 1)$. The equation $f_2 = 0$ yields $a^{10} = (1+\pi^{13})^3/(1+\pi^9)$. On the other hand, if the minor $(s+a^4)(a^{12}+d^4 s^2)$ indicated above is 0, this imposes a nontrivial algebraic condition on π. Thus if π is suitably general, the point is a smooth point on Y.

Hence we will suppose that l is nonzero at the point of Y under consideration. Compute the following 2×2 minors in the 4×4 Jacobian matrix.

$$\begin{aligned} J_{1,3;1,2} &= la^4 r \\ J_{1,4;1,2} &= lb^4 r \\ J_{2,3;1,2} &= (f_3 - lc^{12})r \\ J_{2,4;1,2} &= (f_4 - ld^{12})r \\ J_{3,4;3,4} &= ls^2c^2d^2(ac^{10} + bd^{10}), \end{aligned}$$

where $r = a^8c^3 + b^8d^3$.

Suppose P is a point on Y where the Jacobian has rank less than 2. Since $l \neq 0$, from the above minors, we conclude that $r = 0$. Also not both c and d can be zero. Suppose $c = 0, d \neq 0$. Then $r = 0 \Rightarrow b = 0$, and then $f_4 = 0 \Rightarrow d = 0$, a contradiction. Likewise if $d = 0, c \neq 0$. Thus both c and d are nonzero. Hence either

$$s = ad^3 + bc^3 = 0 \quad \text{or} \quad ac^{10} + bd^{10} = 0.$$

If $s = 0$, then the simultaneous equations $s = 0, r = 0$ in (c^3, d^3) have a non-trivial solution, hence the determinant of coefficients $a^9 + b^9 = 0$, whence from f_3, we get the contradiction that $c = 0$.

If $ac^{10} + bd^{10} = 0$ (and $s \neq 0$), the eighth power of this equation together with $r = 0$, has a non-trivial solution for (a^8, b^8) (since otherwise $s = 0$), hence the determinant of coefficients $c^3d^3(c^{77} + d^{77}) = 0$. It follows that we can take $d = 1, c = \eta$ where $\eta^{77} = 1$, while from $ac^{10} + bd^{10} = 0$, we can take $b = \eta^{10}a$, where $a \neq 0$. The equation $f_2 = 0$ establishes an algebraic equation satisfied by this number a, hence all four coordinates are algebraic over $\mathbf{F}_2$. But now $f_1 = 0$ can be used to solve for $l(P) = \eta + \pi^4$ as an algebraic number, and this is a contradiction if π is transcendental or sufficiently general. □

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Dept. of Mathematics, Washington University, Saint Louis, Missouri 63130
E-mail address: kumar@math.wustl.edu

Dept. of Mathematics, Colorado State University, Fort Collins, Colorado 80523
E-mail address: peterson@math.colostate.edu

Dept. of Mathematics, University of Missouri - St. Louis, Saint Louis, Missouri 63121
E-mail address: rao@arch.umsl.edu