Hilbert scheme components in characteristic 2

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Abstract. *We study the deformations of restrictions to* $\mathbf{P}^3$ *and* $\mathbf{P}^4$ *of Tango's rank 2 bundle on* $\mathbf{P}^5$ *(which exists in characteristic 2). Using this, we construct an example of a family of rank two bundles on* $\mathbf{P}^3$ *(in characteristic 2) with changing* α*-invariant and an example of a component of the Hilbert scheme of smooth surfaces in* $\mathbf{P}^4$ *which exists in characteristic 2 but not in any other characteristic.*

The first author constructed a number of rank two bundles on $\mathbf{P}^4$ in finite characteristics in [M]. Following these ideas, we study a fairly simple rank two bundle in characteristic 2 on $\mathbf{P}^4$ which is stable, with $c_1 = 0, c_2 = 3$. We will explicitly describe the bundle by means of a monad, and we will show that any deformation of this bundle has to have a similar monad. Thus we are considering a component of the moduli space of stable rank two bundles. We find that these bundles cannot exist in characteristic different from two (a result of Barth and Elencwajg [B-E] says such bundles cannot exist in characteristic zero.) It turns out that bundles with such a monad, when restricted to hyperplanes, give a family of rank two bundles on $\mathbf{P}^3$ for which the α-invariant [A-R] is not constant. This is, of course, impossible in characteristics different from two [H]. Hence we have a component of the moduli space of bundles on $\mathbf{P}^4$ (defined over $\mathbf{Z}$) which is supported over the prime 2.

We next look at the surfaces obtained from high twists of these bundles. They give rise to a component of the Hilbert scheme of smooth surfaces in $\mathbf{P}^4$, in characteristic 2. We show that this is a component of the Hilbert scheme of surfaces in $\mathbf{P}^4$ which does not lift to a component of the Hilbert scheme in characteristic zero. Thus we have a component of the Hilbert scheme of surfaces in $\mathbf{P}^4$ (defined over $\mathbf{Z}$) which is supported over the prime 2. In [E-H] (Conjecture C), a similar example is sought: is there a component of the Hilbert scheme of curves in $\mathbf{P}^3$ (defined over $\mathbf{Z}$) which is supported over a prime p?

It turns out that the bundles we construct are quite familiar. We had reached them by a bootstrapping process from $\mathbf{P}^3$. They also happen to be restrictions to $\mathbf{P}^4$ of Tango's rank two bundle on $\mathbf{P}^5$ (in characteristic two). We investigate this connection. It follows that our examples of components of the moduli space and Hilbert scheme on $\mathbf{P}^4$ can be imitated to give similar examples on $\mathbf{P}^5$ of (for example) a component of the Hilbert scheme of smooth threefolds which is not liftable to characteristic zero.

At the end, we discuss examples of non-liftable abstract smooth varieties.

Construction of the bundles:

Claim: In characteristic 2, there is a family of stable rank two bundles $\mathcal{E}$ on $\mathbf{P}^4$, with $c_1 = 0, c_2 = 3$, and with middle cohomology module $H^2_*(\mathcal{E})$ equal to (after twist by 3) a complete intersection module $S/(a,b,c,d,e^2)$, for some choice of basis a,b,c,d,e of linear forms in the polynomial ring S.

We will give a computationally transparent construction of our rank two bundle. Consider the Koszul complex on the five forms a,b,c,d,e^2 where a,b,c,d,e is a basis of linear forms in S, the polynomial ring in 5 variables. The second syzygy bundle $\mathcal{S}$ of the corresponding complex of free sheaves on $\mathbf{P}^4$ appears as the image of the Koszul map

$$4\mathcal{O}_{\mathbf{P}^4}(-3) \oplus 6\mathcal{O}_{\mathbf{P}^4}(-4) \xrightarrow{\Phi} 6\mathcal{O}_{\mathbf{P}^4}(-2) \oplus 4\mathcal{O}_{\mathbf{P}^4}(-3).$$

A matrix for Φ appears below, where we choose the standard exterior algebra basis vectors $e_i \wedge e_j \wedge e_k, (i < j < k)$ and $e_i \wedge e_j (i < j)$ to represent the map, ordering these vectors lexicographically upon the indices. Φ is found using the usual convention: for example, $e_1 \wedge e_2 \wedge e_5 \to ae_2 \wedge e_5 - be_1 \wedge e_5 + e^2 e_1 \wedge e_2$ and is represented by a summand $\mathcal{O}_{\mathbf{P}^4}(-4)$ above and column 3 in the matrix below.

$$\Phi = \begin{bmatrix} c & d & e^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b & 0 & 0 & d & e^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b & 0 & -c & 0 & e^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -b & 0 & -c & -d & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & d & e^2 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & -c & 0 & e^2 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & -c & -d & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 & 0 & e^2 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & b & 0 & -d \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & b & c \end{bmatrix}.$$

Make the following (homogeneous) row and column operations: replace column 5 by (column 5 + b. column 2 + d. column 7) and replace column 10 by (column 10 + column 3);

replace row 1 by (row 1 – row 8) and row 6 by (row 6 – d. row 4 – b. row 9); the resulting matrix Ψ has rows 1 and 6 and columns 5 and 10 given by

$$\Psi = \begin{bmatrix} c & d & e^2 & -a & 0 & 0 & -b & 0 & 0 & 0 \\ & & & & e^2 & & & & & 0 \\ & & & & -b^2 & & & & & 0 \\ & & & & -c & & & & & -b \\ & & & & d^2 & & & & & 0 \\ 0 & a & bd & 0 & 0 & d^2 & -c & -b^2 & e^2 & 2bd \\ & & & & 0 & & & & & a \\ & & & & bd & & & & & e^2 \\ & & & & a & & & & & -d \\ & & & & 0 & & & & & c \end{bmatrix}$$

When the field has characteristic two, the entry $2bd$ becomes zero and using these summands for domain and codomain we end up with a complex

$$2\mathcal{O}_{\mathbf{P}^4}(-4) \to \mathcal{S} \to 2\mathcal{O}_{\mathbf{P}^4}(-2).$$

It is evident from inspecting the two-by-two minors of the two rows and the two columns that this complex is actually a monad giving a rank two bundle. We twist up by 3, let $\mathcal{F} = \mathcal{S}(3)$ and let $\mathcal{E}$ be the rank two bundle with monad

$$(*) \qquad 0 \to 2\mathcal{O}_{\mathbf{P}^4}(-1) \to \mathcal{F} \to 2\mathcal{O}_{\mathbf{P}^4}(1) \to 0.$$

It is an easy computation that $c_1 = 0, c_2 = 3$. The finite length module $H^2_*(\mathcal{E})$ equals $k[a,b,c,d,e]/(a,b,c,d,e^2)$ (shifted in degree by 3.) From the monad, we will now check that $H^0(\mathcal{E}) = 0$.

Let $\mathcal{G}$ be the kernel : $\mathcal{G} \to \mathcal{F} \to 2\mathcal{O}_{\mathbf{P}^4}(1)$. We need to check that $H^1(\mathcal{G}) = 0$. $\mathcal{F}$ has 4 sections corresponding to columns 1,2,4 and 7 of the modified matrix Ψ. It may be noticed that no non-trivial scalar linear combination of these columns can possibly have its first row element zero. So no non-zero section of $\mathcal{F}$ maps to zero in $2\mathcal{O}_{\mathbf{P}^4}(1)$. Hence $H^0(\mathcal{E}) = 0$.

In fact, the stability of $\mathcal{E}$ follows just from the form of the monad. For $h^1(\mathcal{E}(-2)) = 0$ from (*). If $h = 0$ is a general $\mathbf{P}^3$, the structure of $H^2_*(\mathcal{E})$ shows us that the restricted bundle $\mathcal{E}_h$ has $h^1(\mathcal{E}_h(-2)) = 0$. Hence $\mathcal{E}_h$ is stable ([R]) and thus also $\mathcal{E}$.

Deformations of these bundles:

Claim: Let $\mathcal{E}$ be a bundle with the properties described above, existing possibly in any characteristic. Any deformation $\mathcal{E}_t$ of the bundle has the same properties, and in fact has a monad of the same form as (*) above.

Suppose $\mathcal{E}_t$ is a flat family of bundles over $\mathbf{P}^4$ with $\mathcal{E}_0 = \mathcal{E}$. (This family could be either a family in the same characteristic, so that the base space is say a smooth curve over a field, or a family lifting $\mathcal{E}$ from finite characteristic to characteristic zero so that the base space is the spectrum of an integral domain whose fraction field has characteristic zero and which contains a prime ideal with residue field of the given finite characteristic.) We will use the phrase $t \neq 0$ to indicate the generic point of the base. By upper semicontinuity, $\mathcal{E}_t$ is stable for $t \neq 0$ with the same Chern classes.

Also by upper semicontinuity, for $t \neq 0$, $\mathcal{E}_t$ has middle cohomology module $H^2_*(\mathcal{E}_t)$ of three possible types: 0 or k or $k[a,b,c,d,e]/(a,b,c,d,e^2)$ (for some choice of linear forms a,b,c,d,e).

i) Suppose $\mathcal{E}_t$ has middle cohomology module equal to zero. Applying Riemann-Roch, we find that $\mathcal{E}_t(-2)$ has Euler characteristic equal to 1. Hence either $H^0(\mathcal{E}_t(-2))$ or $H^4(\mathcal{E}_t(-2))$ is not equal to zero, and either of these will violate the stability of $\mathcal{E}_t$.

ii) If $\mathcal{E}_t$ has middle cohomology module equal to k, we see from Serre duality that such a rank two bundle must have odd first Chern class. But this is contradictory to our situation.

iii) This last case will take up the rest of this section. We will show that in this case $\mathcal{E}_t$ has a minimal monad exactly like (*), with different maps involved, but with the same bundles. Thus $\mathcal{E}_t$ is of the same type as the $\mathcal{E}$ we started with.

$\mathcal{E}_t$ has middle cohomology module of the same type as $\mathcal{E}_0$ itself (though the forms need not be the same). We will show that any such $\mathcal{E}_t$ fits into a monad of the same type as the monad we constructed for $\mathcal{E}_0$. This will show that our construction gives all such bundles if we let the maps in the monad vary and hence we have described a full component of the moduli space of stable bundles on $\mathbf{P}^4$ with $c_1 = 0, c_2 = 3$.

So let us assume that we have a stable bundle $\mathcal{E}$ with $c_1 = 0, c_2 = 3$ and with $H^2_*(\mathcal{E}) = S/(a,b,c,d,e^2)(3)$. Again using Riemann-Roch, we get $H^1(\mathcal{E}(-2)) = H^1(\mathcal{E}(-3)) = 0$.

Claim: $H^1(\mathcal{E}(-r)) = 0$ for all $r \geq 2$.

In fact, restrict $\mathcal{E}$ to any hyperplane $h = 0$. From the restriction exact sequence, we get $\mathcal{E}_h$ on $\mathbf{P}^3$ such that $H^0(\mathcal{E}_h(-1)) = 0$. It follows that $H^0(\mathcal{E}_h(-r)) = 0$ for all $r \geq 1$, and this implies our claim by focussing on a socle element for $H^1_*(\mathcal{E})$ in degree less than -2, if it were to exist.

We will now go through the monad construction for $\mathcal{E}$ and find that we do get a monad of the type earlier constructed.

The monad construction on $\mathbf{P}^4$ involves 'killing' in turn H^1_* and H^3_* [B-H]. By Horrocks' theorem [Ho1], the middle bundle will be $\mathcal{F}$ plus some line bundles, where $\mathcal{F}$ is the second syzygy bundle for the module $H^2_*(\mathcal{E})$. Since $c_1 = 0$, the monad is self dual. Hence the sum of the line bundles in the middle is also self dual. We conclude that $\mathcal{E}$ has a minimal monad of the form

$$0 \to \oplus\mathcal{O}_{\mathbf{P}^4}(-a_i) \xrightarrow{\alpha} \oplus\mathcal{O}_{\mathbf{P}^4}(-b_i) \oplus \mathcal{F} \oplus \oplus\mathcal{O}_{\mathbf{P}^4}(b_i) \xrightarrow{\beta} \oplus\mathcal{O}_{\mathbf{P}^4}(a_i) \to 0.$$

Here we can assume that each $b_i \geq 0$. By the claim on $H^1_*(\mathcal{E})$ proved above, we can also assume that each $a_i \leq 1$. The minimality means that the $\oplus\mathcal{O}_{\mathbf{P}^4}(a_i)$ picks out minimal generators for the module $H^1_*(\mathcal{E})$. Now, if some $b_i > 0$, the term $\mathcal{O}_{\mathbf{P}^4}(b_i)$ is not contained in the image of α by minimality, and it cannot map to zero by β for this would induce a section of $\mathcal{E}(-b_i)$, contradicting stability. Hence if some $b_i > 0$, then some $a_i > 1$. This shows that all the b_i's must be 0.

Now $\mathcal{F}$ has no sections in degree -1 or less. If some $a_i < 0$, the term $\mathcal{O}_{\mathbf{P}^4}(-a_i)$ must inject into $\mathcal{F}$. This is impossible. Likewise, if some $a_i = 0$, the term $\mathcal{O}_{\mathbf{P}^4}$ must inject into $\mathcal{F}$, (by minimality, it cannot map in a non-zero fashion to any of the $\mathcal{O}_{\mathbf{P}^4}(b_i)$'s). This injection is an injection of bundles, and we inspect columns 1,2,4 and 7 of our original matrix Φ to see that no combination can yield a bundle injection of $\mathcal{O}_{\mathbf{P}^4}$ into $\mathcal{F}$, since the variable e^2 does not appear! Hence all the a_i's are ones. If there are t terms $\oplus\mathcal{O}_{\mathbf{P}^4}(1)$ in the monad, computing Chern classes for $\mathcal{E}$ shows that t must be 2. Hence only our original monad is possible.

Non-existence of these bundles in characteristic $\neq 2$:

Barth and Elencwajg have proved that no stable rank two bundle on $\mathbf{P}^4$ with Chern classes (0,3) can exist in characteristic zero. Using the constancy of the α-invariant on families of rank two bundles on $\mathbf{P}^3$, we show that bundles $\mathcal{E}$ as constructed above in characteristic two cannot exist in characteristics different from two. The argument is simple. Restricting our $\mathcal{E}$ on $\mathbf{P}^4$ to a general hyperplane $h = 0$ gives us a mathematical instanton $\mathcal{E}_h$ on $\mathbf{P}^3$ with $c_1 = 0, c_2 = 3$. The α-invariant $[h^0(\mathcal{E}_h(-2)) - h^1(\mathcal{E}_h(-2))](mod\ 2)$ of such a bundle is 0. The restriction to a special hyperplane yields a rank two bundle on $\mathbf{P}^3$ for which the α-invariant is non-zero. Atiyah and Rees [A-R] proved that this phenomenon cannot occur in characteristic zero for a continuous family of rank two bundles on $\mathbf{P}^3$. Hartshorne [H] extended the proof to any characteristic $\neq 2$. Hence our claim of non-existence.

We also end up with an example, in characteristic 2, of a connected family of (stable) rank two bundles on $\mathbf{P}^3$, with $c_1 = 0, c_2 = 3$ on which the α-invariant is not constant. In fact, we get a family of mathematical instantons degenerating to a bundle with α-invariant non-zero. Thus Hartshorne's result is exact: there is a continuous family of rank two bundles on $\mathbf{P}^3$ in characteristic two for which the α-invariant is non-constant.

For the calculations, consider any bundle with a monad of the type we described in (*). Restrict $\mathcal{E}$ to a general hyperplane $h = 0$ in $\mathbf{P}^4$. Let $\mathcal{E}_h$ be the restricted bundle. The multiplication map by h on the module $S/(a, b, c, d, e^2)$ is an isomorphism between the two non-zero graded pieces. Hence, using the restriction exact sequence, we see that $H^1(\mathcal{E}_h(-2)) = 0$; therefore, without further ado [R], $\mathcal{E}_h$ is a mathematical instanton bundle with $c_1 = 0, c_2 = 3$. In particular, it is stable, hence its α-invariant is 0.

On the other hand consider the restriction $\mathcal{E}_a$ to a special hyperplane, for example $a = 0$. The restriction exact sequence now gives us: $h^1(\mathcal{E}_a(-2)) = 1$ and $h^0(\mathcal{E}_a(-2)) = 0$. Hence the α-invariant of $\mathcal{E}_a$ is 1.

A Hilbert scheme component lying over the prime 2.

Claim: In characteristic 2, there is a component of the Hilbert scheme of smooth surfaces in $\mathbf{P}^4$, with the general surface in the family not liftable to an embedded surface in $\mathbf{P}^4$ in characteristic zero.

Take the bundles in this component of the moduli space in characteristic 2. For a very large twist, we get a family of surfaces, from the zero schemes of sections of these bundles. The general such surface is smooth and subcanonical, with a resolution

$$0 \to \mathcal{O}_{\mathbf{P}^4} \to \mathcal{E}(\nu) \to \mathcal{I}_Y(2\nu) \to 0.$$

Then $\omega_Y = \mathcal{O}_Y(2\nu - 5)$. Hence the line bundle $\omega_Y(5 - 2\nu)$ has $h^0 = 1$ and $H^1 = H^2(\mathcal{I}_Y) = 0$ for large ν.

Now let Y_t be a deformation of Y either in characteristic 2 or to characteristic 0. For $t \neq 0$, Y_t is still smooth. By upper semicontinuity and constancy of the Euler characteristic of $\omega_{Y_t}(5 - 2\nu)$, $h^0(\omega_Y(5 - 2\nu))$ is constant and equal to one. Hence $\omega_{Y_t} = \mathcal{O}_{Y_t}(2\nu - 5)$ for $t \neq 0$ as well. Therefore, Y_t is subcanonical for all t, and comes from a rank two bundle $\mathcal{E}_t$. The Chern classes of $\mathcal{E}_t$ are the same as those of $\mathcal{E}$, and $\mathcal{E}_t$ is stable by upper semi-continuity. From the deformation theory of the bundles $\mathcal{E}$, the bundle $\mathcal{E}_t$ is of the same type as $\mathcal{E}$.

We draw two conclusions: in characteristic 2, Y_t for $t \neq 0$ is of the same type as Y, ie. we have not left our family of surfaces. Hence we have described a full component of the Hilbert scheme of surfaces in characteristic 2. On the other hand, the surface Y is not liftable to characteristic 0 since the bundle itself is not liftable to characteristic zero. Hence there is no component of the Hilbert scheme in characteristic zero which corresponds to the component in characteristic 2.

Relationship of the bundle with Tango's rank two bundle:

Claim: Any extension of a bundle on $\mathbf{P}^4$ with monad of type (*) to $\mathbf{P}^5$ is Tango's rank two bundle and any restriction of Tango's rank 2 bundle on $\mathbf{P}^5$ to $\mathbf{P}^4$ is a bundle of type (*).

We had originally obtained our bundle by building up from information on $\mathbf{P}^3$ (following the methods of [M]). We then realized that this bundle is a restriction to $\mathbf{P}^4$ of Tango's rank two bundle on $\mathbf{P}^5$ in characteristic 2. This bundle was described in [T] in a computational fashion, and Horrocks [Ho2] presented a more representation-theoretic description of the bundle. We will indicate why our bundle on $\mathbf{P}^4$ is indeed a restriction of Tango's rank two bundle.

Suppose $\mathcal{E}$ on $\mathbf{P}^4$, as constructed in the first part, extends to a rank two bundle $\mathcal{F}$ on $\mathbf{P}^5$. Consider the restriction exact sequence. Since $\mathcal{E}$ is stable, $H^0(\mathcal{F}) = 0$. $H^2_*(\mathcal{E})$ is two dimensional with its generator in degree -3. It fits into an exact sequence

$$0 \to \text{cokernel}[H^2_*(h)] \to H^2_*(\mathcal{E}) \to \text{kernel}[H^3_*(h)] \to 0,$$

where $h : \mathcal{F}(-1) \to \mathcal{F}$ is multiplication by h, the equation of $\mathbf{P}^4$ in $\mathbf{P}^5$. If cokernel$[H^2_*(h)] = 0$, then by Nakayama's lemma, $H^2_*(\mathcal{F}) = 0$, and hence $H^3_*(\mathcal{F}) = 0$, by Serre duality, leading to a contradiction that $H^2_*(\mathcal{E}) = 0$. Likewise, kernel$[H^3_*(h)] \neq 0$, either. Therefore, cokernel$[H^2_*(h)]$ must be one-dimensional, situated in degree -2 (since it cannot include the generator of $H^2_*(\mathcal{E})$). Thus $H^2_*(\mathcal{F})$ is a module with one generator, in degree -2.

We now claim that $H^2_*(\mathcal{F})$ is one dimensional, supported in degree -2. For suppose $t \in H^2(\mathcal{F}(-2+p))$ is the lowest degree element annihilated by h. There is then an element $\bar{t} \in H^1(\mathcal{E}(-1+p))$ mapping to t. Since we have found the lowest such degree for such a t, every element of lower degree in $H^1_*(\mathcal{E})$ maps to zero in $H^2_*(\mathcal{F})$. In particular, $\bar{t}$ is a minimal generator of $H^1_*(\mathcal{E})$. But $H^1_*(\mathcal{E})$ has exactly two generators, both in degree -1. Hence $p = 0$. Thus $H^2_*(\mathcal{F})$ is annihilated by h, and since its image in $H^2_*(\mathcal{E})$ is annihilated by all the remaining linear forms, $H^2_*(\mathcal{F})$ is annihilated by all linear forms.

By Serre duality, $H^3_*(\mathcal{F})$ equals k (located in degree -4).

Since $H^1_*(\mathcal{E})$ has two generators, both in degree -1, and one of these is obtained from kernel$H^2_*(h)$, we conclude that $H^1_*(\mathcal{F})$ has one generator in degree -1 and none in lower degrees.

At this point, we can consider the Beilinson square for $\mathcal{F}$. It has the entries:

0	0	0	0	0	0	h^5
1	0	0	0	0	0	h^4
0	1	0	0	0	0	h^3
0	0	0	1	0	0	h^2
0	0	0	0	1	m	h^1
0	0	0	0	0	0	h^0
-5	-4	-3	-2	-1	0	

The value of $m = H^1(\mathcal{F})$ is 7, as will be seen shortly. Using Tango's explicit map to the Grassmannian [T], we may use a computer program like Macaulay to compute the Chern classes and cohomology groups of his bundle, and they agree with those of our postulated extension in this range. Likewise, Horrocks in his version of this bundle [Ho2] computes the invariants of his rank two bundle on $\mathbf{P}^5$, and they are identical to the values in the above Beilinson square.

Now, knowing the numbers just given, we may consider the Beilinson spectral sequence [O-S-S] for $\mathcal{F}$. This yields a monad on $\mathbf{P}^5$ for $\mathcal{F}$ of the following type:

$$0 \to \Omega^5(5) \oplus \Omega^4(4) \xrightarrow{\lambda} \Omega^2(2) \oplus \Omega^1(1) \xrightarrow{\mu} m\mathcal{O}_{\mathbf{P}^5} \to 0.$$

By rank considerations, m must be 7 as indicated above. In this sense, our postulated extension, Tango's bundle and Horrocks' version all have a monad of the above type. In fact, in [D-M-S], there is an explicit example for the matrices λ, μ in the monad above, giving such a bundle in characteristic 2. We will loosely call a bundle with such a monad by the name Tango's bundle.

To complete the discussion, now look at such an $\mathcal{F}$ satisfying a monad of the above form. From the monad, we see that $\mathcal{F}$ is stable, and $H^1_*(\mathcal{F})$ has one generator in degree -1 and the rest in degree 0. We will let $\mathcal{E}_4$ be the restriction of $\mathcal{F}$ to any hyperplane $\mathbf{P}^4$: $h = 0$. It is clear from the restriction sequence that $H^2_*(\mathcal{E}_4)$ is 2 dimensional, supported in degrees -3 and -2. We will argue that the module structure is isomorphic to $S/(a, b, c, d, e^2)$ and that $\mathcal{E}_4$ is stable. Then our earlier discussion shows that $\mathcal{E}_4$ has monad of type (*).

Let $t \in H^1(\mathcal{F}(-1))$ be non-zero. Observe that it is not possible for two linear forms h, g to annihilate t in $H^1(\mathcal{F})$. For let $\mathcal{E}_3$ be the restriction of $\mathcal{F}$ to the $\mathbf{P}^3$: $h = g = 0$. It is easy to check that $\mathcal{E}_3$ gets 2 minimal sections in degree 0 which is certainly not correct.

Returning to $\mathcal{E}_4$, either it is stable $(h.t \neq 0)$ with $h^1(\mathcal{E}_4) = 6$ or it is semistable with $h^0(\mathcal{E}_4) = 1$ and $h^1(\mathcal{E}_4) = 7$. In either case, our observation on annihilators of t tells us that $H^1_*(\mathcal{E}_4)$ has at most 4 generators, 2 in degree -1 and possibly 2 more in degree 0. Hence if we kill H^1_* and H^3_* to build a monad for $\mathcal{E}_4$ on $\mathbf{P}^4$, the middle bundle has rank at most 10. Now it is evident that the possibility that $H^2_*(\mathcal{E}_4) = k(3) \oplus k(2)$ is ruled out by this, for its second syzygy module has rank 12 which is greater than 10. Hence the module structure of $H^2_*(\mathcal{E}_4)$ is established.

We next claim that $\mathcal{E}_4$ is stable (so no h can annihilate t). To see this, take a general restriction of $\mathcal{E}_4$ to $\mathcal{E}_3$ on $\mathbf{P}^3$. Since $h^1(\mathcal{E}_4(-2)) = 0$, from the known multiplicative structure of $H^2_*(\mathcal{E}_4)$, we conclude that $H^1(\mathcal{E}_3(-2)) = 0$. By [R], $\mathcal{E}_3$ is stable, hence also $\mathcal{E}_4$.

Remarks:

Ellia and Hartshorne in [E-H] raise the question of whether there is a family of curves in $\mathbf{P}^3$ in finite characteristic which does not lift to a family of curves in $\mathbf{P}^3$ in characteristic zero. They also ask if there is a full family of curves (ie. a component of the Hilbert scheme) in $\mathbf{P}^3$ in finite characteristic which does not lift to a family of curves in $\mathbf{P}^3$ in characteristic zero. This is the embedded version of the lifting question for varieties. It is known from deformation theory [G] that any smooth projective curve in characteristic p can be lifted to characteristic zero (the obstruction to deformations lies in H^2 of the tangent sheaf). Various examples are known of smooth varieties in characteristic p which do not lift to characteristic zero (starting with [S]). One method is to find examples which violate Kodaira's vanishing theorem. We give an example of the argument.

Consider the variety X defined in characteristic p in [L-R]. It carries a very ample line bundle L and does not satisfy Kodaira's vanishing theorem with this line bundle. It follows from the proof of Kodaira's vanishing theorem given by Deligne-Illusie [D-I] that the pair (X, L) does not lift to characteristic zero. Now suppose the abstract variety X lifts to characteristic zero. The deformation theory of the line bundle L says that if the obstruction space $H^2(X, \mathcal{O}_X)$ equals zero, then the line bundle L can be lifted to characteristic zero along with X [G]. This will be a contradiction to the statement that (X, L) does not lift. In this example, X is obtained from $\mathbf{P}^3$ by successively projectivizing vector bundles, and so the condition $H^2(X, \mathcal{O}_X) = 0$ boils down to the fact that $H^2(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}) = 0$. In conclusion, the variety X does not lift to characteristic zero.

(This example also gives a weak version of Hartshorne's question for higher dimensional embedded varieties. The pair (X, L) gives an embedding of X in some $\mathbf{P}^n$. This embedding does not lift to characteristic zero. To strengthen this example, one would have to show that X and its translates by automorphisms of $\mathbf{P}^n$ give a component of the Hilbert scheme.)

The bundles we construct in this paper also give examples of non-liftable varieties. Let $X = \mathbf{P}(\mathcal{E})$ with $X \xrightarrow{\pi} \mathbf{P}^4$. Let $L = \mathcal{O}_\pi(1)$ be the tautological line bundle on X, and let M be the pull-back of $\mathcal{O}_{\mathbf{P}^4}(1)$. Suppose X lifts to X_t in characteristic zero. Since

$H^2(X, \mathcal{O}_X) = 0$, L and M also lift along with X. Since M has no higher cohomology, the global sections of M also lift and this yields a morphism $X_t \xrightarrow{\pi_t} \mathbf{P}^4_t$. Again, since L has no higher direct images on $\mathbf{P}^4$, $\pi_{t*}L_t$ is a rank two bundle on $\mathbf{P}^4_t$. This contradicts the fact that our bundle $\mathcal{E}$ does not lift to characteristic zero. Of course, this particular example is not as general as the first type of example, since it works only for characteristic 2.

It is interesting to note the ideas underlying these examples of non-liftable varieties. Serre's original example uses the fact that certain finite groups can exist in the general linear group over a field of characteristic p, and not in characteristic zero, and he constructs quotients by such groups. The other technique was to use the fact that Kodaira vanishing is false in characteristic p. The example we use above in characteristic 2 depends upon the non-constancy of the α-invariant. This in turn depends ultimately upon the special behaviour of quadratic forms in characteristic 2.

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