Vector Bundles on Projective Spaces

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Abstract. We generalise a criterion of G. Kempf for a vector bundle on projective space to be decomposable. We also give a similar criterion for a vector bundle to be homogeneous.

G. Kempf had proved the following splitting criterion for vector bundles over $\mathbb{P}^n$ [**Kem90**].

THEOREM 1 (Kempf). *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ for $n \geq 2$. Then $\mathcal{E}$ is a direct sum of line bundles if and only if $\mathcal{E}$ satisfies the following two conditions.*

(1) $\mathrm{H}^1(\mathbb{P}^n, \mathcal{E}\mathrm{nd}\,\mathcal{E}(-e)) = 0\,\forall e > 0$.
(2) *$\mathcal{E}$ extends to a vector bundle on $\mathbb{P}^{n+1}$.*

The aim of this note is to generalise his criterion. We show that his condition of extending the vector bundle to $\mathbb{P}^{n+1}$ is not necessary. We also prove a similar criterion for a vector bundle to be homogeneous.

THEOREM 2. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ for $n \geq 2$. Then $\mathcal{E}$ is a direct sum of line bundles if and only if*

$$\mathrm{H}^1(\mathbb{P}^n, \mathcal{E}\mathrm{nd}\,\mathcal{E}(-e)) = 0\,\forall e > 0.$$

PROOF. By the theorem of Kempf [**Kem90**], we need only show that $\mathcal{E}$ extends to $\mathbb{P}^{n+1}$.

We proceed as follows. Fix an embedding of $H = \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$. If $p \in \mathbb{P}^{n+1}$ and $p \notin H$, let $U_p = \mathbb{P}^{n+1} - \{p\}$ and let $\pi_p : U_p \to H$ be the natural projection, which is identity on H. Let $\mathcal{E}_p = \pi_p^* \mathcal{E}$. We will actually show that $\mathcal{E}_p$ extends as a vector bundle to the whole of $\mathbb{P}^{n+1}$. As one can easily see, this implies the decomposabilty of $\mathcal{E}$ even without appealing to the above criterion of Kempf.

Let $q \neq p$ be any other point as above- that is, $q \notin H$. Then we have a vector bundle $\mathcal{E}_q$ on U_q. Let $U = U_p \cap U_q$. We have $\mathcal{E}_{p|H} \cong \mathcal{E}_{q|H} \cong \mathcal{E}$, by our choice. I claim that $\mathcal{E}_{p|mH} \cong \mathcal{E}_{q|mH}$ for any integer $m > 0$, where, as usual mH denotes the m^{th} order thickening of H in $\mathbb{P}^{n+1}$. By our choice, we have this for $m = 1$. So, assume that the result is proved for some m and let us verify this for $m + 1$. We have an exact sequence,

$$0 \to \mathcal{E}(-m) = \mathcal{E}_{p|H}(-m) \to \mathcal{E}_{p|(m+1)H} \to \mathcal{E}_{p|mH} \to 0.$$

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Homing this with $\mathcal{E}_q$, we get,

$$\mathrm{Hom}\,(\mathcal{E}_q, \mathcal{E}_{p|(m+1)H}) \to \mathrm{Hom}\,(\mathcal{E}_q, \mathcal{E}_{p|mH}) \to \mathrm{H}^1(\mathcal{E}\mathrm{nd}\,\mathcal{E}(-m)) = 0.$$

By induction, we have a surjective homomorphism in the middle term above and since the next term is zero, we can lift it to an element in the first term, which is just a homomorphism $\mathcal{E}_q \to \mathcal{E}_{p|(m+1)H}$. It is clear that this homomorphism is surjective and we have proved our induction step.

Thus we see that $\hat{\mathcal{E}_p} \cong \hat{\mathcal{E}_q}$ where ˆ denotes the formal completion of U along H (see [**Kem90**]). Now by Grothendieck's Lefschetz theory [**Gro68**], one sees that there exists a Zariski open neighbourhood V of H in U where $\mathcal{E}_p$ and $\mathcal{E}_q$ are isomorphic. Since V is an open neighbourhood of H in $\mathbb{P}^{n+1}$ and H is ample, $V = \mathbb{P}^{n+1} - \{p_1, \ldots, p_m\}$ for a finite set of points p_i. We may without loss of generality assume that the p_i's are distinct and $p = p_1, q = p_2$. Let $U' = \mathbb{P}^{n+1} - \{p_2, \ldots, p_m\} = U_q - \{p_3, \ldots, p_m\}$. Then $U_p \cup U' = \mathbb{P}^{n+1}$ and $U_p \cap U' = V$. The vector bundles $\mathcal{E}_p$ on U_p and $\mathcal{E}_{q|U'}$ on U' are isomorphic on the intersection V and hence they patch up to give a vector bundle on the whole of $\mathbb{P}^{n+1}$, which is clearly an extension of $\mathcal{E}$. □

Next we prove a similar theorem for homogeneity.

THEOREM 3. *A vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ is homogeneous if either*

$$\mathrm{H}^1(\mathbb{P}^n, \mathcal{E}\mathrm{nd}\,\mathcal{E}) \text{ or } \mathrm{H}^1(\mathbb{P}^n, \mathcal{E}\mathrm{nd}\,\mathcal{E}(-1))$$

is zero.

PROOF. Let me first look at the case when $\mathrm{H}^1(\mathcal{E}\mathrm{nd}\,\mathcal{E}(-1)) = 0$. Let $H \subset \mathbb{P}^n$ be a hyperplane and let G_H be the group of automorphisms of $\mathbb{P}^n$ which fix H and acts as identity on H. As H varies, these subgroups generate $\mathrm{Aut}\,\mathbb{P}^n$ $(\mathrm{SL}(r,k) = \mathrm{E}(r,k))$ and since the set of elements $g \in \mathrm{Aut}\,\mathbb{P}^n$ such that $g^*\mathcal{E} \cong \mathcal{E}$ form a group, it suffices to show that $\mathcal{E}$ is invariant under elements of $G_H = G$ for a fixed H.

We have maps, $m : \mathbb{P}^n \times G \to \mathbb{P}^n$, the action of the group, $p : \mathbb{P}^n \times G \to \mathbb{P}^n$, the first projection and $q : \mathbb{P}^n \times G \to G$, the second projection. Consider the vector bundle, $\mathcal{F} = m^*\mathcal{E}\check{} \otimes p^*\mathcal{E}(-1)$ on $\mathbb{P}^n \times G$, where ˇ denotes dual. By our hypothesis, we have $\mathrm{H}^1(\mathbb{P}^n \times \{e\}, \mathcal{F}_{|\mathbb{P}^n \times \{e\}}) = 0$ where $e \in G$ is the identity element. Thus by semicontinuity, for g in an open neighbourhood U of e, we get $\mathrm{H}^1(\mathbb{P}^n \times \{g\}, \mathcal{F}_{|\mathbb{P}^n \times \{g\}}) = 0$. That is, $\mathrm{H}^1(\mathbb{P}^n, g^*\mathcal{E}\check{} \otimes \mathcal{E}(-1)) = 0$. From the exact sequence,

$$0 \to g^*\mathcal{E}\check{} \otimes \mathcal{E}(-1) \to g^*\mathcal{E}\check{} \otimes \mathcal{E} \to \mathcal{E}\check{} \otimes \mathcal{E}_{|H} \to 0,$$

by taking global sections, we get that the identity element in the last term can be lifted to a homomorphism $g^*\mathcal{E} \to \mathcal{E}$. Since, this homomorphism restricted to H is an isomorphism, we see that this map is generically surjective and since it is a map between vector bundles of equal rank, we see that this map must be an injection of sheaves. But, they have the same determinant and so, this map must be an isomorphism. So, we have shown that for all $g \in U$, $g^*\mathcal{E} \cong \mathcal{E}$. Again, since this set is a subgroup of G, we see that $g^*\mathcal{E} \cong \mathcal{E}$ for all $g \in G$. This finishes the proof.

Next, let us assume that $\mathrm{H}^1(\mathcal{E}\mathrm{nd}\,\mathcal{E}) = 0$. This case seems to be well known among experts, but let me sketch a proof here. In characteristic zero, the fact that the above H^1 is the tangent space for the deformation space of $\mathcal{E}$ will finish the proof of homogeneity. But, this does not really need characteristic zero and the proof is similar to the one above. So, let $G = \operatorname{Aut}\mathbb{P}^n$ and let m, p, q be as above. Again, consider the sheaf $\mathcal{F} = p^*\mathcal{E}\check{}\otimes m^*\mathcal{E}$. Then, the hypotheis implies that $\mathrm{H}^1(\mathcal{F}_{|\mathbb{P}^n\times\{e\}}) = 0$ and so, by semicontinuity, there exists an open affine neighbourhood U of e such that $R^1q_*\mathcal{F} = 0$ on U. We have an exact sequence,

$$0 \to \mathfrak{m} \to \mathcal{O}_G \to k = k(e) \to 0$$

where $\mathfrak{m}$ is the ideal sheaf defining the identity on G. Tensoring with $\mathcal{F}$, we have,

$$0 \to q^*\mathfrak{m} \otimes \mathcal{F} \to \mathcal{F} \to \mathcal{E}\mathrm{nd}\,\mathcal{E} \to 0.$$

By taking direct image with respect to q on U, we have,

$$q_*\mathcal{F} \to q_*\mathcal{E}\mathrm{nd}\,\mathcal{E} = \mathrm{H}^0(\mathcal{E}\mathrm{nd}\,\mathcal{E}) \to R^1q_*(q^*\mathfrak{m} \otimes \mathcal{F}) = 0,$$

the last zero by the vanishing of $R^1q_*\mathcal{F}$ and projection formula. Since U is affine, $q_*\mathcal{F} = \mathrm{H}^0(\mathcal{F})$, on $\mathbb{P}^n \times U$. Thus, lifting the identity element of $\mathrm{H}^0(\mathcal{E}\mathrm{nd}\,\mathcal{E})$ to $q_*\mathcal{F} = \mathrm{H}^0(\mathcal{F})$, we get a homomorphism $f : p^*\mathcal{E} \to m^*\mathcal{E}$, which when restricted to $\mathbb{P}^n \times \{e\}$ is an isomorphism. Let $Z \subset \mathbb{P}^n \times U$ be the support of the cokernel of f. Then, since $Z \cap \mathbb{P}^n \times \{e\}$ is empty and q is proper, we get that $q(Z) \subset U$ is a proper closed subset, not containing e. Thus, replacing U by $V = U - Z$, we may assume that f is in fact surjective on $\mathbb{P}^n \times V$. Since f is a map between two vector bundles of the same rank, we see that, then f must be an isomorphism on $\mathbb{P}^n \times V$. But, this implies, for any $g \in V$, $g^*\mathcal{E} \cong \mathcal{E}$. But, again, the set of $g \in G$ with this property is a subgroup and since it contains the non-empty open subset V, this set must be all of G. Thus, $\mathcal{E}$ is homogeneous. □

References

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