Submitted to Contemporary Mathematics

# Standard vector bundle deformations on $\mathbf{P}^n$

N. Mohan Kumar, Chris Peterson, and A. Prabhakar Rao

ABSTRACT. This paper illustrates how certain construction methods for low rank vector bundles on projective spaces can be modified to produce interesting deformations of bundles.

## 1. Introduction

The problem of constructing enough vector bundles on projective spaces is an old one. The case of rank one vector bundles (line bundles) on $\mathbf{P}^n$ is well understood. The only line bundles on $\mathbf{P}^n$ are the twists (or powers), $\mathcal{O}_{\mathbf{P}}(\nu)$, of the hyperplane bundle $\mathcal{O}_{\mathbf{P}}(1)$. The case of rank $r$ vector bundles on $\mathbf{P}^n$ with $r > 1$ is not very well understood. Of course, one can obtain some rank $r$ bundles by taking the direct sum of bundles of rank less than $r$. This, in general, yields only a small subset of the set of all bundles of rank $r$. A direct sum of line bundles will be called a free bundle, and a direct sum of bundles, all equal to $\mathcal{O}_{\mathbf{P}}$ will be called a trivial bundle. We will be concerned with the problem of producing non-free bundles. For any free bundle $\mathcal{G}$ on $\mathbf{P}^n$, it is true that the cohomology modules $H^i_*(\mathbf{P}^n, \mathcal{G})$ (defined as $\oplus_{\mu\in\mathbb{Z}} H^i(\mathbf{P}^n, \mathcal{G}(\mu))$, where $\mathcal{G}(\mu)$ is the twist $\mathcal{G} \otimes \mathcal{O}_{\mathbf{P}}(\mu)$ of $\mathcal{G}$), are zero for $1 \le i \le n-1$. A theorem of Horrocks asserts the converse. This combination of facts will provide a useful tool for establishing that a bundle is not free.

It is easy to obtain non-free bundles on $\mathbf{P}^n$ if the rank of the bundle is allowed to be large enough. By a theorem of Serre, given any bundle $\mathcal{G}$ on $\mathbf{P}^n$, of rank $m$, where $m > n$, then for $\mu$ large enough, the twist $\mathcal{G}(\mu)$ contains a trivial subbundle of rank $m - n$. The quotient will be non-free because of the following argument that we insert here (since it will subsequently appear often). The dual map $\mathcal{G}^\vee(-\mu) \to (m-n)\mathcal{O}_{\mathbf{P}} \to 0$ is not surjective on global sections since when $\mu$ is large, $\mathcal{G}^\vee(-\mu)$ will have no global sections while clearly, $(m-n)\mathcal{O}_{\mathbf{P}}$ has $m-n$ global sections. Hence the bundle which is the kernel, call it $\mathcal{K}$, has $H^1_*(\mathbf{P}^n, \mathcal{K}) \neq 0$ by the long exact sequence of cohomology. The bundle, $\mathcal{K}$, is therefore not free, and neither is the quotient which is its dual, $\mathcal{K}^*$.

Since in this argument, $\mathcal{G}$ could have been a free bundle, there is no obstruction to the construction of non-free bundles of rank $n$ (or more) on $\mathbf{P}^n$. The problem

1991 *Mathematics Subject Classification.* Primary 14F05, 13D02, 14J60, 32L05, 13A35.
*Key words and phrases.* Vector Bundles, Deformations.
The authors acknowledge support from the NSF during this work.

hence is really one of constructing non-free bundles of rank less than $n$ on $\mathbf{P}^n$. We will indicate by *corank* the difference between the dimension of the space and the rank of the bundle. Hence one looks for non-free bundles on $\mathbf{P}^n$ of positive corank. The case of $\mathbf{P}^2$ is trivial, and corank one bundles on $\mathbf{P}^3$ have been well studied, with many non-free bundles of rank two, both stable and unstable, available in large families.

The outstanding question in this area is whether there exists on $\mathbf{P}^6$ a rank two bundle which is not a sum of two line bundles. This question was posed by Mumford and Horrocks when they constructed the rank two Horrocks-Mumford bundle on $\mathbf{P}^4$. In characteristic zero, the Horrocks-Mumford bundle is the only known rank two bundle on $\mathbf{P}^4$ which is not free, except for minor variations on it like the pull-back via morphisms from $\mathbf{P}^4$ to $\mathbf{P}^4$. In rank three, there are bundles on $\mathbf{P}^4$ of at least two types known: a construction due to Trautmann and Vetter and independently by Tango, and a construction of Sasakura. On $\mathbf{P}^5$, there is a rank three bundle that was constructed by Horrocks called the parent bundle, and there are variations of this bundle. Apart from the Horrocks-Mumford bundle and Horrocks' parent bundle (together with their modifications), in characteristic zero, all other known positive corank, non-free bundles are of corank one. There are two main methods for the construction of corank one bundles on $\mathbf{P}^n$. One construction gives null-correlation bundles and more generally mathematical instanton bundles on $\mathbf{P}^n$ where $n$ is odd. The other construction, of Trautmann-Vetter and also Tango, gives corank one bundles on $\mathbf{P}^n$ for any $n$. No corank two bundles are known on $\mathbf{P}^n$ (with $n \geq 6$) in any characteristic.

Mohan Kumar in [**7**], [**8**] discussed the construction of bundles on $\mathbf{P}^n$ as well as the construction of bundles on $\mathbf{P}^{n-1} \times \mathbb{A}^1$. In his discussion, he addressed a problem (due to Peskine) about the existence of a family of rank two bundles on $\mathbf{P}^3$, where the general bundle is free and the special one is non-free. Both this question and the question of the existence of non-free rank two bundles on $\mathbf{P}^4$ (other than the Horrocks-Mumford bundle) were translated into one of constructing certain bundles on $\mathbf{P}^3$ with nilpotent endomorphisms. In positive characteristic, Mohan Kumar, through this translation, succeeded in obtaining examples for each of the two questions, in much the same way. In addition, in any characteristic, his approach gives rank three bundles on $\mathbf{P}^4$ as well as families of rank three bundles on $\mathbf{P}^3$, where the general bundle is free and the special one is non-free. These were also studied in [**10**] and [**11**].

In view of this, in this paper, we look at Peskine's question in higher rank, on $\mathbf{P}^n$. The question is whether there are families of bundles on $\mathbf{P}^n$, where the general one is free, and the special one is not. Once again, as in the construction of bundles on $\mathbf{P}^n$, this question is not of interest if the rank is too large. For example, the bundle of 1-forms on $\mathbf{P}^n$ fits into the Euler exact sequence on $\mathbf{P}^n$

$$0 \to \Omega^1_{\mathbf{P}} \to (n+1)\mathcal{O}_{\mathbf{P}}(-1) \to \mathcal{O}_{\mathbf{P}} \to 0.$$

Since this is a non-split sequence, we immediately conclude that the family parametrized by $\mathbb{A}^1 = \mathrm{Ext}^1(\mathcal{O}_{\mathbf{P}}, \Omega^1_{\mathbf{P}})$ has generic member equal to $(n+1)\mathcal{O}_{\mathbf{P}}(-1)$ and special member (when the extension class is zero) equal to $\Omega^1_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}$.

Hence the Peskine question is interesting only when we consider families of bundles on $\mathbf{P}^n$ of corank greater than or equal to zero. Our purpose in this paper is to restrict ourselves to arbitrary characteristic, and review the constructions of some of the low rank and low corank bundles on $\mathbf{P}^n$ listed above. The ones that we review allow also a modified construction that creates a bundle of the same rank, not on $\mathbf{P}^n$, but on $\mathbf{P}^{n-1} \times \mathbb{A}^1$. Our modification is quite an easy one. The bundles we review are all constructed out of the standard Koszul complex on $\mathbf{P}^n$, which is the complex built out of a regular sequence of $n+1$ linear forms on $\mathbf{P}^n$. We modify the constructions to include also regular sequences on $\mathbf{P}^{n-1} \times \mathbb{A}^1$, where the degrees of the terms in the regular sequence can vary, with one term being the zero degree affine parameter. This modification, which is straightforward, has the pleasant consequence that since, when the affine parameter is non-zero, the restriction of the Koszul complex to the $\mathbf{P}^{n-1}$ fibre splits, we get a family of bundles for which the general bundle is free but the special bundle is non-free.

Our results are as follows:

(1) a family of rank three bundles on $\mathbf{P}^3$, where the general bundle is free (and non-trivial) while the special bundle is non-free.
(2) a family of rank three bundles on $\mathbf{P}^3$, where the general bundle is trivial while the special bundle is non-free.
(3) a family of rank three bundles on $\mathbf{P}^4$, where the general bundle is free (and either trivial or non-trivial) while the special bundle is non-free.
(4) a family of corank zero bundles on $\mathbf{P}^n$ ($n$ even), where the general bundle is free (and either trivial or non-trivial) while the special bundle is non-free.
(5) a family of corank zero bundles on $\mathbf{P}^n$ ($n$ odd, $n \geq 5$), where the general bundle is free (and non-trivial) while the special bundle is non-free.

Missing from our results are:

- a family of corank zero bundles on $\mathbf{P}^n$ ($n$ odd, $n \geq 5$), where the general bundle is trivial while the special bundle is non-free. From our point of view, the case $n = 3$ is special because we can exploit many different constructions on $\mathbf{P}^4$, that of Trautmann-Vetter and that of Sasakura, as well as the Horrocks construction on $\mathbf{P}^5$ (which can be restricted to $\mathbf{P}^4$) and the method of Mohan Kumar. For higher even dimensional spaces, we have only the construction of Trautmann-Vetter to exploit.
- a family of rank two bundles on $\mathbf{P}^3$, as in Peskine's original question, in zero characteristic. From our point of view, in characteristic zero, there is only one known rank two bundle on $\mathbf{P}^4$, and it resists the kind of exploitation that worked for our other cases. G. Ottaviani has pointed out an explanation for this resistance. The modified constructions we make below are similar to Horrocks' notion of a weighted pull-back which can be created for bundles with a $\mathbf{C}^*$ group of symmetries. The Horrocks-Mumford bundle has only a discrete group of symmetries.
- on $\mathbf{P}^3$, and given three degrees, a family of bundles on $\mathbf{P}^3$ where the general bundle is a sum of line bundles of those degrees while the special one is non-free. Both Trautmann-Vetter and Horrocks constructions give arithmetic sequences for the degrees, while Sasakura's construction gives even degrees.

The authors would like to thank G. Ottaviani for correspondence on these topics. The third author would like to thank Jun-Muk Hwang for an invitation to KIAS, and for his enquiry about deforming rank three bundles which started this work, and also Zhenbo Qin for an invitation to the conference on Hilbert Schemes,

Vector Bundles and their interplay with Representation Theory, where this work was presented.

## 2. The Koszul complex

Let $X$ represent either $\mathbf{P}^n$ or $\mathbf{P}^{n-1} \times \mathbb{A}^1$ over a field $k$. We will denote by $S_X$ the graded polynomial ring which in the first case is $k[X_0, X_1, \dots, X_n]$ (where the $X_i$'s of degree one give a basis of linear forms on $\mathbf{P}^n$) and in the second case is $k[X_0, X_1, \dots, X_{n-1}, t]$ (where the $X_i$'s of degree one give a basis of linear forms on $\mathbf{P}^{n-1}$ and $t$ of degree zero is the parameter on $\mathbb{A}^1 = \operatorname{Spec} k[t]$).

We will let $\mathcal{O}_X(\nu)$ denote the power of the hyperplane line bundle on $\mathbf{P}^n$ or the pull-back of $\mathcal{O}_X(\nu)$ on $\mathbf{P}^{n-1}$. A global section of $\mathcal{O}_X(\nu)$ will be a form in $S_X$ of degree $\nu$. For a bundle $\mathcal{E}$ on $X$, we denote by $H^i_*(\mathcal{E})$ the graded $S_X$ module $\bigoplus_{\nu\in\mathbb{Z}} H^i(X, \mathcal{E}(\nu))$ .

Let us establish notation for Koszul complexes.

Let $a_0, a_1, \dots, a_r$ be a sequence of homogeneous forms on $X$. By this we mean a sequence of homogeneous elements chosen from $S_X$. Let $d_i$ equal the degree of $a_i$. Let $K_1 = \oplus_{i=0}^r e_i \mathcal{O}_X(-d_i)$, where $e_0, e_1, \dots, e_r$ is a basis for the direct sum. Let $K_q = \wedge^q K_1$. We denote $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_q}$ by $e_{i_1 i_2 \dots i_q}$ for short and when $i_1 < i_2 < \dots i_q$, these will be our basis vectors of choice for $K_q$, appearing with $\mathcal{O}_X(-d_{i_1} - d_{i_2} - \cdots - d_{i_q})$. The Koszul map from $K_{q+1}$ to $K_q$ is given by $\sum a_i e_i^*$, where $\{e_i^*\}$ is the dual basis to $\{e_i\}$. We will use a matrix representation of the Koszul map, where the bases of $K_{q+1}$ and $K_q$ are ordered lexicographically by multi-index, calling the matrix $M_q$. So for example, $e_{13}$ is mapped to $a_3 e_1 - a_1 e_3$ , and if $r = 4$, then the sixth column of $M_1$ is $\begin{bmatrix} 0 & a_3 & 0 & -a_1 & 0 \end{bmatrix}^\vee$.

The resulting complex $(K_\bullet, M_\bullet)$ is called the Koszul complex on $X$ corresponding to the sequence $a_0, a_1, \dots, a_r$. We denote by $\mathcal{K}^q$ the image of the Koszul map given by $M_q$.

There are three cases we will be interested in.

- If $X = \mathbf{P}^n$ and the sequence is $a_0, a_1, \dots, a_n$, a regular sequence of $n+1$ homogeneous forms (necessarily of positive degree) on $X$, then the Koszul complex is exact, and the image $\mathcal{K}^q$ of the Koszul map $M_q$ will be a vector sub-bundle of $K_q$ of rank $\binom{n}{q}$. $\mathcal{K}^q$ is generated by its global sections (in various twists) $e_{i_1 i_2 \dots i_{q+1}}$. Each $\mathcal{K}^q$ is non-free, for $0 < q < n$.
- If $X = \mathbf{P}^n$ and the sequence is $a_0, a_1, \dots, a_{n+1}$ where $a_0, a_1, \dots, a_n$ is a regular sequence of $n+1$ homogeneous forms on $X$ and $a_{n+1}$ is the unit element 1, then the Koszul complex is split exact, and each $\mathcal{K}^q$ is a direct summand of $K_q$, hence free.
- If $X = \mathbf{P}^n$ and the sequence is $a_0, a_1, \dots, a_{n+1}$ where $a_0, a_1, \dots, a_n$ is a regular sequence of $n+1$ homogeneous forms on $X$ and $a_{n+1}$ is the zero element, then the Koszul complex is the direct sum of two copies of the Koszul complex on $a_0, a_1, \dots, a_n$, where the second is shifted in grading by one. In particular, the Koszul complex is exact and $\mathcal{K}^q$ is a vector bundle which is non-free in the range $0 < q < n+1$.
- The *standard Koszul complex* is when $X = \mathbf{P}^n$ and the regular sequence is $\{l_0, l_1, \dots, l_n\}$, where these are independent linear forms on $\mathbf{P}^n$. The bundles $\mathcal{K}^q$ are then the usual bundles of $q$-forms $\Omega^q$ on $\mathbf{P}^n$.

When we are looking for deformations, we will be considering the case $X = \mathbf{P}^n \times \mathbb{A}^1$, and deal with regular sequences of the form $a_0, a_1, \ldots, a_{n+1}$ where $a_{n+1} = t$ is the affine parameter of degree 0 and $a_0, a_1, \ldots, a_n$ is a regular sequence on $\mathbf{P}^n$, not depending on $t$. This will give rise to bundles $\mathcal{K}^q$ on $\mathbf{P}^n \times \mathbb{A}^1$, which will be viewed as a family of bundles on the fibres $\mathbf{P}^n$ of the product. When we look at a fibre $\mathbf{P}^n$ over a nonzero value, say 1, of $t$, the restriction of $\mathcal{K}^q$ will be a sum of line bundles, since the Koszul complex will split. But the restriction to the special fibre $t = 0$ will be non-split. (When $q = 1$, this re-creates the banal example obtained from the Euler sequence that we described in the introduction.)

## 3. Null-correlation bundles.

Let $X = \mathbf{P}^n$ be odd dimensional, with $n \geq 3$. Let $S_X = k[X_0, X_1, \ldots, X_n]$ be the underlying polynomial ring. Choose a regular sequence of linear forms $l_0, l_1, \ldots, l_n$. Then the following is the *monad* for a rank $n-1$ bundle, $\mathcal{N}$, on $\mathbf{P}^n$:

$$0 \to \mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{[-l_n, +l_{n-1}, -l_{n-2}, \ldots, -l_1, +l_0]^\vee} (n+1)\mathcal{O}_{\mathbf{P}} \xrightarrow{[l_0, l_1, \ldots l_{n-1}, l_n]} \mathcal{O}_{\mathbf{P}}(1) \to 0.$$

In other words,

- this sequence is a complex (obvious from the matrices of the maps),
- the left hand map is an injection of vector bundles (since the map is never zero at any point of $\mathbf{P}^n$),
- the right hand map is a surjection of vector bundles (since again the map is never zero at any point of $\mathbf{P}^n$).

Hence the homology at the middle is a vector bundle, $\mathcal{N}$, of rank $n-1$. (The oddness of $n$ is required because this monad is pairing the linear forms.) The bundle is non-free because the right hand map is not surjective on the level of global sections in degree $-1$, and hence the $S$-module $H^1_*(\mathbf{P}^n, \mathcal{N})$ is non-zero and Horrocks' theorem applies.

This construction is easily mimicked over $X = \mathbf{P}^n \times \mathbb{A}^1$, when $\mathbf{P}^n$ is *even* dimensional. Let $S_X = k[X_0, X_1, \ldots, X_n, t]$ be the underlying polynomial ring of $X$. Let $a_0, a_1, \ldots, a_n, t$ be a regular sequence in $S_X$ of homogeneous forms of degree $n+1, n, n-1, \ldots, 1, 0$, where $a_0, a_1, \ldots, a_n$ are forms on $\mathbf{P}^n$ itself, independent of $t$. We construct a monad on $X$ as follows:

$$(3.1) \qquad 0 \to \mathcal{O}_X(-\frac{n+1}{2}) \xrightarrow{[-t, +a_n, -a_{n-1}, \ldots, +a_0]^\vee} \oplus_{i=0}^{n+1} \mathcal{O}_X(-\frac{n+1}{2} + i) \xrightarrow{[a_0, a_1, \ldots, a_n, t]} \mathcal{O}_X(\frac{n+1}{2}) \to 0.$$

Our choice of degrees makes this a valid complex. By inspection of the ranks of the two matrices at each point of $X$, we see that this is the monad of a vector bundle $\mathcal{N}$ on $\mathbf{P}^n \times \mathbb{A}^1$, of rank $n$. Consider the general member of this family of rank $n$ bundles on $\mathbf{P}^n$, when $\mathbf{P}^n$ is the fibre over a point $t \neq 0$. Both the left hand map and the right hand map of the monad are easily seen to be split in this case. Hence for $t \neq 0$, say $t = 1$, $\mathcal{N}_{t=1}$ is the sum of line bundles $\oplus_{i=1}^n \mathcal{O}_{\mathbf{P}}(-\frac{n+1}{2} + i)$. On the other hand, when $t = 0$, the monad is a minimal monad on $\mathbf{P}^n$, and our earlier argument, showing that the null-corellation bundle is non-free, is equally valid in showing that $\mathcal{N}_{t=0}$ is not free.

This monad is so simple that we can alter degrees of the homogeneous elements to some extent. Here is another choice: choose $a_1, \ldots, a_{n-1}$ all of the same degree, equal to $\frac{n+1}{2}$, and $a_0$ of degree $n+1$ as before. Now the monad looks like

$$0 \to \mathcal{O}_X(-\frac{n+1}{2}) \xrightarrow{[-t,+a_n,-a_{n-1},\dots,+a_0]^\vee} \tag{3.2}$$
$$\mathcal{O}_X(-\frac{n+1}{2}) \oplus (n-1)\mathcal{O}_X \oplus \mathcal{O}_X(\frac{n+1}{2}) \xrightarrow{[a_0,a_1,\dots,a_n,t]} \mathcal{O}_X(\frac{n+1}{2}) \to 0.$$

It follows in this case that the general fibre $\mathcal{N}_{t=1} = (n-1)\mathcal{O}_{\mathbf{P}}$, a trivial bundle. So, on even dimensional projective spaces, we can find families of corank zero bundles where the special bundle is non-free and the general bundle is free, or even trivial. In fact, after normalizing, we can make the general fibre to be anything we want as long as the degrees of the summands are symmetric around zero. However, this construction does not yield free general fibres where, for example, the first Chern class is odd.

## 4. Trautmann-Vetter bundles.

The construction of null-correlation bundles, described above, works only on odd dimensional projective spaces. A construction due to Trautmann-Vetter and, independently by Tango, creates bundles of rank $n-1$ on any dimensional projective space. Tango's approach is to perform Chern class calculations on bundles appearing in the standard Koszul complex, which we cannot easily mimick in our setting for deformations. However, Trautmann-Vetter give a more explicit construction in terms of the bases and matrices of the Koszul complex and this readily carries over to the setting for deformations, provided some care is taken to ensure homogeneity of maps. (Cascini [**3**] has also made variations of these bundles to construct 'weighted Tango bundles'.)

The construction of the Trautmann-Vetter bundle is achieved by considering global sections of $\Omega^1$ on $\mathbf{P}^n$. Using the regular sequence $X_0, X_1, \dots, X_n$ on $\mathbf{P}^n$, and the standard Koszul complex $K_\bullet$ built out of it, $\Omega^1(2)$ has a basis of $\binom{n+1}{2}$ global sections given by $e_{ij}$ (where $i<j$) which generate the bundle at each point of $\mathbf{P}^n$. However, Trautmann-Vetter observe that just $2n-1$ sections suffice to generate the bundle $\Omega^1(2)$ at each point, given by the following set:

$$\{ \sum_{\substack{i<j \\ i+j=k}} e_{ij} | k = 1, 2, \dots, 2n-1 \}.$$

So for example, on $\mathbf{P}^3$, five sections $e_{01},\ e_{02},\ e_{03}+e_{12},\ e_{13},\ e_{23}$ generate $\Omega^1(2)$. The proof is obvious when we look at the representation of these sections under the inclusion $\Omega^1 \subseteq K_1$. For example, in the case of $\mathbf{P}^3$, we get the 5 columns of the matrix

$$\begin{bmatrix} X_1 & X_2 & X_3 & 0 & 0 \\ -X_0 & 0 & X_2 & X_3 & 0 \\ 0 & -X_0 & -X_1 & 0 & X_3 \\ 0 & 0 & -X_0 & -X_1 & -X_2 \end{bmatrix}.$$

Now, to verify that these 5 sections generate the rank 3 bundle $\Omega^1(2)$ on $\mathbf{P}^3$ at each point, it is enough to show that some $3 \times 3$ minor of the matrix is non-vanishing at each point of $\mathbf{P}^3$. But it is evident that modulo $X_0, X_1, \dots, X_{i-1}$, a $3 \times 3$ upper triangular matrix can be extracted with $X_i$ along the diagonal. This argument clearly also works in the case of general $n$, giving Trautmann-Vetter's observation.

Let $\mathcal{T}$ be the kernel of the induced surjection:

$$0 \to \mathcal{T} \to \oplus_{k=1}^{2n-1}\{\sum_{\substack{i<j\\ i+j=k}} e_{ij}\}\mathcal{O}_{\mathbf{P}}(-2) \to \Omega^1 \to 0.$$

In this case, $\mathcal{T}$ is a rank $n-1$ bundle on $\mathbf{P}^n$, and is not a sum of line-bundles since the map to $\Omega^1(2)$ does not surject onto all the global sections, hence $H^1(\mathbf{P}^n, \mathcal{T}(-2)) \neq 0$.

With this example to follow, we have a Trautmann-Vetter type construction of a family of rank $n$ bundles on $\mathbf{P}^n$, where the generic bundle is a sum of line bundles and the special one is not, obtained as follows.

First choose a regular sequence $a_0, a_1, \ldots, a_n, a_{n+1} = t$ on $\mathbf{P}^n \times \mathbb{A}^1$, where $a_0, a_1, \ldots, a_n$ is a regular sequence on $\mathbf{P}^n$, independent of $t$, with degree $a_i$ equal to $d_i = n+1-i$. Consider the Koszul complex $K_\bullet$ on $\mathbf{P}^n \times \mathbb{A}^1$ given by this sequence, and consider $\mathcal{K}^1$. Each sum, $\sum_{\substack{i<j\\ i+j=k}} e_{ij}$, is a *homogeneous* section of $\mathcal{K}^1(k)$, for $1 \leq k \leq 2n+1$. These $2n+1$ sections generate the rank $n+1$ bundle $\mathcal{K}^1$, by an argument based upon matrices that is identical to the argument above. The kernel is therefore a bundle $\mathcal{T}$ on $X = \mathbf{P}^n \times \mathbb{A}^1$, of rank $n$, where

$$0 \to \mathcal{T} \to \oplus_{k=1}^{2n+1}\{\sum_{\substack{i<j\\ i+j=k}} e_{ij}\}\mathcal{O}_X(-k) \to \mathcal{K}^1 \to 0.$$

When $t \neq 0$, say $t = 1$, the restriction to the fibre $\mathbf{P}^n$ is trivial because $\mathcal{K}^1_{t=1}$ is trivial on this fibre and the matrix of the representation of the $2n+1$ sections shows explicitly a splitting because of the uppermost diagonal with $n+1$ copies of $t$. When $t = 0$, the restriction $\mathcal{K}^1_{t=0}$ has a summand of $\mathcal{O}_{\mathbf{P}}$, yet the $2n+1$ sections we have chosen do not map onto the generating section of this $\mathcal{O}_{\mathbf{P}}$. Hence $\mathcal{T}_{t=0}$ picks up a non-zero element in $H^1_*$ from the exact sequence connecting $\mathcal{T}_{t=0}$ and $\mathcal{K}^1_{t=0}$. Hence, by Horrocks' theorem, $\mathcal{T}_{t=0}$ is non-free.

The general bundle in this case is seen to be $\oplus_{k=n+2}^{2n+1}\mathcal{O}_{\mathbf{P}}(-k)$. So we have obtained a family of rank $n$ bundles on $\mathbf{P}^n$ where the general bundle is free (but not trivial), and the special bundle is not free. The combinatorics of Trautmann-Vetter's construction forces the degrees of the general bundles to be always in a non-trivial arithmetic sequence (provided $X = \mathbf{P}^n \times \mathbb{A}^1$ has dimension $\geq 4$). Hence the types of degrees allowed for the general bundle in the family are quite restricted. In particular, it does not provide an example where the general bundle is trivial.

## 5. Sasakura's bundle

The Trautmann-Vetter construction above gives a family of rank 3 bundles on $\mathbf{P}^3$, where the general bundle is free, and the special bundle is not. After normalizing, the general fibre is of the form $\mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(1) \oplus \mathcal{O}_{\mathbf{P}}(2)$. Clearly, by pull-backs via maps from $\mathbf{P}^4$ to $\mathbf{P}^4$, one can arrange the general fibre to be of the form $\mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(a) \oplus \mathcal{O}_{\mathbf{P}}(2a)$, *ie.* any arithmetic progression of degrees is achievable. To get examples where the general fibre is not of this form, a different rank three bundle contruction on $\mathbf{P}^4$ due to Sasakura can be used. A treatment of this construction can be found in [**1**]. The treatment described below is analogous to a discussion in [**9**].

The construction of Sasakura's rank three bundle is also based, like Trautmann-Vetter's bundle, on a Koszul complex on $\mathbf{P}^4$, though not the standard one. The difference is that while the Trautmann-Vetter rank three bundle appears as the kernel of a surjection

$$7\mathcal{O}_{\mathbf{P}}(-2) \to \Omega^1_{\mathbf{P}^4},$$

Sasakura's bundle appears as the homology of a monad of the form

$$0 \to 2\mathcal{O}_{\mathbf{P}} \to 9\mathcal{O}_{\mathbf{P}} \to \mathcal{K}^1 \to 0.$$

where we do not specify here the regular sequence, nor the degrees in the monad above.

In what follows, $X$ will be either $\mathbf{P}^4$ or $\mathbf{P}^3 \times \mathbb{A}^1$, and we will take a regular sequence $a_0, a_1, a_2, a_3, a_4$ on $X$. (In a change from previous examples, when $X = \mathbf{P}^3 \times \mathbb{A}^1$, it will be $a_2$ which will be $t$, and the others a regular sequence on $\mathbf{P}^3$.)

In the Koszul complex on the above regular sequence, subject to the conditions of homogeneity, replace the ten basis elements of $K_2$ by a new basis where $e_{04}$ and $e_{13}$ are replaced by $e_{04} + e_{13}, e_{04} - e_{13}$. $\mathcal{K}^1$ is generated by the nine basis elements after leaving out $e_{04} - e_{13}$ (this follows for instance, from the stronger observation of Trautmann-Vetter). Since $\mathcal{K}^1$ has rank four, the kernel of this surjection by nine copies of $\mathcal{O}_X$ has rank five.

Consider the basis vectors of $K_3$ consisting of $e_{024} + e_{123}, e_{234} + a_1 e_{013} - a_0 e_{014}$, once again subject to conditions of homogeneity. It is an easy calculation that these two vectors map to the subspace given by the nine generating sections, hence are maps to the rank five kernel. Their $9 \times 2$ matrix in this basis is of rank two at each point of the space. These follow from

$$\begin{aligned}
M_2(e_{024} + e_{123}) &= a_0 e_{24} - a_2 e_{04} + a_4 e_{02} + a_1 e_{23} - a_2 e_{13} + a_3 e_{12} \\
&= -a_2(e_{04} + e_{13}) + a_0 e_{24} + a_1 e_{23} + a_4 e_{02} + a_3 e_{12} \\
M_2(e_{234} + a_1 e_{013} - a_0 e_{014}) &= a_2 e_{34} - a_3 e_{24} + a_4 e_{23} + a_1 a_0 e_{13} - a_1^2 e_{03} \\
&+ a_1 a_3 e_{01} - a_0^2 e_{14} + a_0 a_1 e_{04} - a_0 a_4 e_{01} \\
&= a_0 a_1 (e_{04} + e_{13}) - a_3 e_{24} + a_4 e_{23} + a_2 e_{34} - a_1^2 e_{03} \\
&+ a_1 a_3 e_{01} - a_0^2 e_{14} - a_0 a_4 e_{01}.
\end{aligned} \tag{5.1}$$

So indeed we have a rank two sub-bundle of the rank 5 kernel of the surjection $9\mathcal{O} \to \mathcal{K}^1$, delivering our rank three bundle $\mathcal{S}$ as the quotient.

The homogeneity conditions are as follows:

$$\begin{aligned}
d_0 + d_4 &= d_1 + d_3 \\
d_0 + d_2 + d_4 &= d_1 + d_2 + d_3 \\
d_2 + d_3 + d_4 &= (d_1) + d_0 + d_1 + d_3 = (d_0) + d_0 + d_1 + d_4.
\end{aligned}$$

These work out to $d_0 + d_4 = d_1 + d_3, d_2 + d_4 = d_0 + 2d_1, d_2 + d_3 = 2d_0 + d_1$.

Sasakura's bundle on $X = \mathbf{P}^4$ is obtained if we choose $d_2 = 2$, $d_0 = d_1 = d_3 = d_4 = 1$.

Corollary 5.1. Given even positive integers $2k, 2l$, there exists a family of rank three bundles on $\mathbf{P}^3$ (obtained through Sasakura's construction) such that the special bundle is non-free and the general bundle is $\mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(2k) \oplus \mathcal{O}_{\mathbf{P}}(2l)$.

Proof. We choose on $X = \mathbf{P}^3 \times \mathbb{A}^1$, the regular sequence $a_0, a_1, t, a_3, a_4$ where the $a_i$'s are forms on $\mathbf{P}^3$. We get the monad of a rank three bundle $\mathcal{S}$ on $X$, and when $t = 0$, the restriction $\mathcal{S}_{t=0}$ is non-free as usual.

When $t$ is say equal to 1, the restriction of $\mathcal{K}^1$ splits as $\mathcal{O}_{\mathbf{P}}(-d_0) \oplus \mathcal{O}_{\mathbf{P}}(-d_1) \oplus \mathcal{O}_{\mathbf{P}}(-d_3) \oplus \mathcal{O}_{\mathbf{P}}(-d_4)$, and the monad described above can be inspected to see the explicit splitting

$$\mathcal{S}_{t=1} = \mathcal{O}_{\mathbf{P}}(-d_0 - d_1) \oplus \mathcal{O}_{\mathbf{P}}(-d_0 - d_3) \oplus \mathcal{O}_{\mathbf{P}}(-d_1 - d_4).$$

By the homogeneity conditions, $d_4 = d_0 + 2d_1, d_3 = 2d_0 + d_1$, hence the dual degrees are $d_0 + d_1, 3d_0 + d_1, d_0 + 3d_1$. Twisting down by $d_0 + d_1$ gives the degrees $0, 2d_0, 2d_1$, where $d_0, d_1$ can be freely chosen as any $k, l$. □

## 6. Horrocks Bundles

We will present the rank three bundle on $\mathbf{P}^5$ discovered by Horrocks (called the parent bundle). Variants of this bundle, obtained by altering the degrees of the forms used in the construction, have been studied by Horrocks, and in [**2**] and [**4**]. Our purpose is to display the construction in such a way that the existence of a similar construction on $\mathbf{P}^4 \times \mathbb{A}^1$ becomes quite apparent. The upshot will be a family of rank 3 bundles on $\mathbf{P}^4$, generically a sum of line bundles, with the specialization at $t = 0$ non-split.

Accordingly, let $X$ stand for either $\mathbf{P}^5$ or $\mathbf{P}^4 \times \mathbb{A}^1$. Let $a_0, a_1, a_2, a_3, a_4, a_5$ be a regular sequence on $X$, with $d_i$ equal to the degree of $a_i$. Assume that

$$d_0 + d_5 = d_1 + d_4 = d_2 + d_3 = 2d, \text{ an even number}$$
$$d_0 + d_1 + d_2 = d_3 + d_4 + d_5 = D.$$

Consider the Koszul complex $K_\bullet$ built on this regular sequence. Let $V = K_1(d)$, with basis vectors $e_0, \ldots, e_5$. Its dual $V^\vee$ has a dual basis $e_0^*, \ldots, e_5^*$. The element $\eta = e_0^* \wedge e_5^* + e_1^* \wedge e_4^* + e_2^* \wedge_3^*$, a homogeneous element of degree 0, gives rise to three maps:

$$\mathcal{O}_X \xrightarrow{\eta} \wedge^2 V^\vee$$
$$V \xrightarrow{\eta} V^\vee, \text{skew-symmetric,}$$
$$\wedge^2 V \xrightarrow{\wedge^2 \eta} \wedge^2 V^\vee, \text{symmetric.}$$

Next, let $\mathcal{E} = \mathcal{K}^1(d)$, with $\iota : \mathcal{E} \hookrightarrow V$ the inclusion. The induced map

$$\mathcal{E} \xrightarrow{\iota^\vee \eta \iota} \mathcal{E}^\vee$$

is skew-symmetric, and corresponds to the map

$$\rho : \mathcal{O}_X \xrightarrow{\eta} \wedge^2 V^\vee \xrightarrow{\wedge^2 \iota^\vee} \wedge^2 \mathcal{E}^\vee.$$

<u>Claim 1</u>: The induced map $\wedge^2 \rho : \wedge^2 \mathcal{E} \to \wedge^2 \mathcal{E}^\vee$ is symmetric, with image $\mathcal{A}$, a vector bundle of rank 6.

The symmetry of the map is obvious. In order to check the rank statement, consider the rank 4 bundle $\mathcal{N}$ obtained from the homology of the monad

$$0 \to \mathcal{O}_X(-d) \xrightarrow{[-a_5, -a_4, -a_3, a_2, a_1, a_0]^\vee} V \xrightarrow{[a_0, a_1, a_2, a_3, a_4, a_5]} \mathcal{O}_X(d) \to 0,$$

hence given as

$$0 \to \mathcal{O}_X(-d) \to \mathcal{E} \to \mathcal{N} \to 0.$$

By direct calculation, under the map $\mathcal{E} \xrightarrow{\iota^\vee \eta \iota} \mathcal{E}^\vee$, the section of $\mathcal{E}$ given by $0 \to \mathcal{O}_X(-d) \to \mathcal{E}$ maps to zero in $\mathcal{E}^\vee$, and likewise, the dual map $\mathcal{E} \to \mathcal{E}^\vee \to \mathcal{O}_X(d)$

is zero. Hence $\mathcal{E} \xrightarrow{\iota^\vee \eta \iota} \mathcal{E}^\vee$ factors through a map of rank 4 bundles $\mathcal{N} \xrightarrow{\psi} \mathcal{N}^\vee$. $\psi$ is forced to be skew-symmetric as well. On the other hand, since $\eta\iota : \mathcal{E} \to V^\vee$ has image a bundle of rank 5, and since $V^\vee \to \mathcal{E}^\vee \to 0$ has kernel of rank 1, the image of $\mathcal{E} \xrightarrow{\iota^\vee \eta \iota} \mathcal{E}^\vee$ has rank at least 4 at each point.

Hence $\mathcal{E} \xrightarrow{\iota^\vee \eta \iota} \mathcal{E}^\vee$ factors through a skew-symmetric isomorphism

$$\psi : \mathcal{N} \to \mathcal{N}^\vee.$$

In particular, $\wedge^2 \rho : \wedge^2 \mathcal{E} \to \wedge^2 \mathcal{E}^\vee$ has image isomorphic to $\wedge^2 \mathcal{N}$, hence $\mathcal{A} = \wedge^2 \mathcal{N}$, of rank 6.

Claim 2: $\mathcal{A} = \wedge^2 \mathcal{N}$ has $\mathcal{O}_X$ as a summand.

In fact, the skew-symmetric isomorphism $\psi : \mathcal{N} \to \mathcal{N}^\vee$ induces a symmetric isomorphism $\wedge^2\psi : \wedge^2\mathcal{N} \to \wedge^2\mathcal{N}^\vee$. Now compose the inverse of this with the map $\psi : \mathcal{O}_X \to \wedge^2\mathcal{N}^\vee$ and its dual. The resulting composite

$$\mathcal{O}_X \to \wedge^2\mathcal{N}^\vee \xrightarrow{(\wedge^2\psi)^{-1}} \wedge^2\mathcal{N} \to \mathcal{O}_X$$

is a constant map since locally, at a point, we may calculate it using the local representation $\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ for $\psi$, and the calculation of the composite is seen to be the multiplication by the constant 2. This shows the splitting of the copy of $\mathcal{O}_X$.

(Note: Certainly, on $X = \mathbf{P}^5$, this shows a splitting. When we deal with the case of $X = \mathbf{P}^4 \times \mathbb{A}^1$, this will induce a map from $\mathcal{O}_X$ to $\mathcal{O}_X$ which is non-zero at each point. This map must also be a constant (independent of $t$).)

Let $\mathcal{S}$ be the kernel of the map $\wedge^2\mathcal{N} \to \mathcal{O}_X$. $\mathcal{S}$ is a vector bundle of rank 5 and

$$\wedge^2\mathcal{N} = \mathcal{S} \oplus \mathcal{O}_X.$$

Consider the basis element $e_{012}$ of $K_3 = (\wedge^3 V)(-3d)$ giving the inclusion $\mathcal{O}_X(-D) \to (\wedge^3 V)(-3d)$. Under the Koszul map $\sum a_i e_i^* : (\wedge^3 V)(-3d) \to (\wedge^2 V)(-2d)$, $e_{012}$ maps to $a_0 e_{12} - a_1 e_{02} + a_2 e_{01}$. Since

$$0 \to \mathcal{E} \xrightarrow{\iota} V \to \mathcal{O}_X(d) \to 0,$$

we have also the exact sequence

$$0 \to \wedge^2\mathcal{E} \xrightarrow{\wedge^2\iota} \wedge^2 V \to V \otimes \mathcal{O}_X(d).$$

In turn, under the map $\wedge^2 V \to V \otimes \mathcal{O}_X(d)$, the image of $e_{012}$ is sent to zero since $e_{12} \mapsto a_1 e_2 - a_2 e_1$ etc.

Hence, under the Koszul map, $e_{012}$ maps to $(\wedge^2\mathcal{E})(-2d) \subseteq (\wedge^2 V)(-2d)$. On the other hand, the composite map $\wedge^2\mathcal{E} \to \wedge^2\mathcal{N} \to \mathcal{O}_X$ factors through $\wedge^2 V \xrightarrow{\eta^\vee} \mathcal{O}_X$. A calculation shows that $e_{012}$ is sent to $a_0\eta^\vee e_{12} - a_1\eta^\vee e_{02} + a_2\eta^\vee e_{01} = 0$ in $\mathcal{O}_X$.

Hence $e_{012} : \mathcal{O}_X(-D + 2d) \to \mathcal{S} \subseteq \wedge^2\mathcal{N}$.

Likewise $e_{345}$.

There is a similar dual picture: the basis element $e_{012}$ also induces $K_3^\vee \to \mathcal{O}_X(D)$. The composite under the dual Koszul map gives

$$(\wedge^2 V^\vee)(2d) \to (\wedge^3 V^\vee)(3d) \to \mathcal{O}_X(D)$$

seen to be given by $a_2e_{01} - a_1e_{02} + a_0e_{12}$. It is an easy check that this composite vanishes on the image of $V^\vee \otimes \mathcal{O}_X(-d)$ in $(\wedge^2 V^\vee)$, giving an induced map

$$\wedge^2\mathcal{E}^\vee \to \mathcal{O}_X(D-2d).$$

Hence there is an induced a map

$$\wedge^2\mathcal{N}^\vee \to \mathcal{O}_X(D-2d).$$

The composite of this map with the inclusion $\mathcal{O}_X \xrightarrow{\psi} \wedge^2\mathcal{N}^\vee$ is calculated via the composite $\mathcal{O}_X \xrightarrow{\psi} \wedge^2V^\vee \to \mathcal{O}_X(D-2d)$, and is easily seen to be zero. Hence this map given by $e_{012}$ from $\wedge^2\mathcal{N}^\vee \to \mathcal{O}_X(D-2d)$ factors through the subset $\mathcal{S} \subseteq \wedge^2\mathcal{N}$.

Likewise $e_{345}$.

Claim 3: The complex

$$0 \to \mathcal{O}_X(-D+2d) \xrightarrow{e_{012}+e_{345}} \mathcal{S} \xrightarrow{e_{012}-e_{345}} \mathcal{O}_X(D-2d) \to 0$$

is a monad of vector bundles.

Viewing the image of $\mathcal{O}_X(-D+2d) \xrightarrow{e_{012}+e_{345}} \mathcal{S}$ as an element $a_0e_{12} - a_1e_{02} + a_2e_{01} + a_3e_{45} - a_4e_{35} + a_5e_{34}$ of $\wedge^2V$, it is evident that the left hand map is an inclusion of bundles. Likewise the right hand map can be viewed as induced from $\wedge^2V^\vee \to \mathcal{O}_X(D-2d)$, and is given by $a_0e_{12} - a_1e_{02} + a_2e_{01} + a_3e_{45} - a_4e_{35} + a_5e_{34}$, hence is a surjection of bundles. The composite is calculated via the isomorphism $\wedge^2\eta : \wedge^2V \to \wedge^2V^\vee$. We get $\wedge^2\eta(a_0e_{12} - a_1e_{02} + a_2e_{01} + a_3e_{45} - a_4e_{35} + a_5e_{34}) = a_0e^*_{43} - a_1e^*_{53} + a_2e^*_{54} + a_3e^*_{10} - a_4e^*_{20} + a_5e^*_{21}$, in $\wedge^2V^\vee$, and hence the final image in $\mathcal{O}_X(D-2d)$ is $-a_0a_5 - a_1a_4 - a_2a_3 + a_3a_2 + a_4a_1 + a_5a_0 = 0$. Thus we have a monad.

The homology of this bundle is called a Horrocks rank 3 bundle. It is quite plainly not a sum of line bundles because the map

$$\mathcal{S} \xrightarrow{e_{012}-e_{345}} \mathcal{O}_X(D-2d) \to 0$$

as calculated above, is not surjective on global sections.

The *parent bundle* on $\mathbf{P}^5$ is obtained if we take the standard Koszul complex on $\mathbf{P}^5$. Variants on $\mathbf{P}^5$ can be obtained with other choices of degrees for the regular sequence.

COROLLARY 6.1. There exist on $\mathbf{P}^4$ families of rank 3 bundles, where the general bundle is a sum of line bundles, and the special one is not so. The general bundle can be chosen as trivial or with degrees in any arithmetic sequence.

PROOF. Perform the construction of a Horrocks rank 3 bundle on $\mathbf{P}^4 \times \mathbb{A}^1$, choosing a sequence $a_0, a_1, a_2, a_3, a_4, t$, where $a_i$ is a form on $\mathbf{P}^4$ of degree $d_i$, and where $d_0 = d_1 + d_4 = d_2 + d_3$ and $d_0 + d_1 + d_2 = d_3 + d_4$. On a fibre $\mathbf{P}^4$ for which $t \neq 0$, the restrictions of $\mathcal{E}$ and $\mathcal{N}$ are split as sums of line bundles. Hence, so are the restrictions of $\mathcal{A} = \wedge^2\mathcal{N}$ and its summand $\mathcal{S}$. The map

$$0 \to \mathcal{O}_X(-D+2d) \xrightarrow{e_{012}+e_{345}} \mathcal{S}$$

as calculated above is split on this fibre because it splits as a map to $\wedge^2V$. Likewise, the map

$$\mathcal{S} \xrightarrow{e_{012}-e_{345}} \mathcal{O}_X(D-2d) \to 0$$

is split on the fibre. The homology of the monad on this fibre can be explicitly calculated as $\mathcal{O}_\mathbf{P}(d_4 - d_3) \oplus \mathcal{O}_\mathbf{P} \oplus \mathcal{O}_\mathbf{P}(d_3 - d_4)$. When $t = 0$, the homology of the monad is a rank three bundle which is not the sum of line bundles for the reason mentioned at the end of the main construction.

Now choose degrees $4, 1, 1, 3, 3$ for $d_0, d_1, d_2, d_3, d_4$ (with $d_5 = 0$). The family obtained has general fibre trivial.

Choose degrees $6, 1, 2, 4, 5$ for $d_0, d_1, d_2, d_3, d_4$ (with $d_5 = 0$). The family obtained has general fibre free but non-trivial. In fact, it is $\mathcal{O}_{\mathbf{P}}(-1) \oplus \mathcal{O}_{\mathbf{P}} \oplus \mathcal{O}_{\mathbf{P}}(1)$, so any non-trivial arithmetic sequence of degrees can be obtained by pull-backs of this example. □

## 7. An example where the general bundle on $\mathbf{P}^3$ is trivial

This last example is different from all the earlier ones in that it does not rely on properties of the Koszul complex. It produces a family of rank three bundles on $\mathbf{P}^3$ where the general bundle is trivial and the special bundle is non-free. Modifications of this method might be the way to show that any three integers are achievable as the degrees of the general bundle in a family on $\mathbf{P}^3$. We have not yet been able to establish if this is true.

On $\mathbf{P}^3$, consider the matrix

$$\varphi = \begin{bmatrix} 0 & -X_0^3 & -X_1^3 & -\alpha \\ X_0^3 & 0 & -\beta & -X_2^6 \\ X_1^3 & \beta & 0 & -X_3^6 \\ \alpha & X_2^6 & X_3^6 & 0 \end{bmatrix},$$

where $\alpha = X_0X_3^2 - X_1X_2^2$ and $\beta = X_0^2X_3^4 + X_0X_1X_2^2X_3^2 + X_1^2X_2^4$ are chosen to make the pfaffian of the matrix vanish. $\varphi$ gives a map from $\mathcal{O}_{\mathbf{P}} \oplus 3\mathcal{O}_{\mathbf{P}}(-3) \to \mathcal{O}_{\mathbf{P}} \oplus 3\mathcal{O}_{\mathbf{P}}(3)$, which is checked to be of rank two at each point on $\mathbf{P}^3$, hence we get its image to be a rank two sub-bundle $A$. A very similar matrix

$$\psi = \begin{bmatrix} 0 & X_3^6 & -X_2^6 & \beta \\ -X_3^6 & 0 & \alpha & -X_1^3 \\ X_2^6 & -\alpha & 0 & X_0^3 \\ -\beta & X_1^3 & -X_0^3 & 0 \end{bmatrix}$$

gives a second rank two bundle $B$ and since $\psi\varphi = 0$, we get the short exact sequence

$$0 \to A \to \mathcal{O}_{\mathbf{P}} \oplus 3\mathcal{O}_{\mathbf{P}}(3) \to B \to 0.$$

$A, B$ are non-free bundles and hence this is a nontrivial extension $\eta \in \mathrm{Ext}^1(B, A)$. By choosing as $\mathbb{A}^1$, the line $\{t\eta | t \in k\}$ spanned by $\eta$ in this vector space, we get a family of rank four bundles on $\mathbf{P}^3$ where the general one is $\mathcal{O}_{\mathbf{P}} \oplus 3\mathcal{O}_{\mathbf{P}}(3)$ and the special one (with $t = 0$) is $A \oplus B$. It is clear that both $A$ and $B$ have a section in degree zero, say $s$ and $s'$, where $s$ maps to the column vector $\begin{bmatrix} 0 & X_0^3 & X_1^3 & \alpha \end{bmatrix}^\vee$, and $s'$ is the image of $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\vee$. Hence the section $\begin{bmatrix} 1 & X_0^3 & X_1^3 & \alpha \end{bmatrix}^\vee$ of the general fibre $\mathcal{O}_{\mathbf{P}} \oplus 3\mathcal{O}_{\mathbf{P}}(3)$ becomes, in the limit, the section $(s, s')$ of $A \oplus B$. It is evident from inspection that $s$ (the first column of $\varphi$) and $s'$ (the first column of $\psi$) do not both vanish at the same point of $\mathbf{P}^3$. Hence we have a nowhere vanishing section of the rank four bundle on $\mathbf{P}^3 \times \mathbb{A}^1$, and the quotient is a rank three bundle on $\mathbf{P}^3 \times \mathbb{A}^1$. When $t \neq 0$, the rank three bundle obtained is $3\mathcal{O}_{\mathbf{P}}(3)$, and when $t = 0$, we get the rank three bundle $(A \oplus B)/(s, s')$, which is clearly non-free.

## References

[1] H. Abo, W. Decker and N. Sasakura, *An elliptic conic bundle in $\mathbf{P}^4$ arising from a stable rank-3 vector bundle*, Mathematische Zeitschrift **229** no.4 (1998), 725–741.

[2] V. Ancona and G. Ottaviani, *The Horrocks bundles of rank three on $P^5$*, J. Reine Angew. Math. **460** (1995), 69–92.

[3] P. Cascini, *Weighted Tango bundles on $P^n$ and their moduli spaces*, Forum Math. **13** (2001), 251–260.

[4] W.Decker, N.Manolache and F.O.Schreyer, *Geometry of the Horrocks bundle on $\mathbf{P}^5$*, Complex Projective Geometry, London Math. Soc. Lecture Note Series 179, 128–148, Cambridge 1992.

[5] G. Horrocks, *Examples of rank three vector bundles on five-dimensional projective space*, Journal of the London Mathematical Society **18** (1978), 15–27.

[6] G. Horrocks and D. Mumford, *A rank 2 vector bundle on $\mathbf{P}^4$ with 15,000 symmetries*, Topology **12** (1973), 63–81.

[7] N. Mohan Kumar, *Smooth Degeneration of Complete Intersection Curves in Positive Characteristic*, Inventiones Mathematicae **104** (1991), 313–319.

[8] N. Mohan Kumar, *Construction of rank two vector bundles on $\mathbf{P}^4$ in positive characteristic*, Inventiones Mathematicae **130** (1997), 277–286.

[9] N. Mohan Kumar, C. Peterson and A.P. Rao, *Hilbert scheme components in characteristic 2*, Communications in Algebra **28** (2000), 5735–5744.

[10] N. Mohan Kumar, C. Peterson and A.P. Rao, *Constructing low rank vector bundles on $\mathbf{P}^4$ and $\mathbf{P}^5$*, J. Algebraic Geometry **11** (2002), 203–217.

[11] N. Mohan Kumar, C. Peterson and A.P. Rao, *Degenerating families of rank two bundles*, preprint.

[12] H. Tango, *An example of indecomposable vector bundle of rank $n-1$ on $\mathbf{P}^n$*, Journal of Mathematics of Kyoto University **16** no. 1 (1976), 137–141.

[13] G. Trautmann, *Darstellung von Vektorraumbündeln über $\mathbf{C^n} - \{\mathbf{0}\}$*, Arch. Math. **24** (1973), 303–313.

[14] U. Vetter, *Zu einem Satz von G. Trautmann über den Rang gewisser kohärenter analytischer Moduln*, Archiv der Mathematik (Basel) **24** (1973), 158- -161.

Dept. of Mathematics, Washington University, Saint Louis, MO 63130
*E-mail address*: `kumar@math.wustl.edu`

Dept. of Mathematics, Colorado State University, Fort Collins, CO 80523
*E-mail address*: `peterson@math.colostate.edu`

Dept. of Mathematics, University of Missouri - St. Louis, Saint Louis, MO 63121
*E-mail address*: `rao@arch.umsl.edu`