REMARKS ON UNIMODULAR ROWS

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ABSTRACT. If (a, b, c) is a unimodular row over a commutative ring A and if the polynomial $z^2 + bz + ac$ has a root in A, we show that the unimodular row is completable. In particular, if $1/2 \in A$ and $b^2 - 4ac$ has a square root in A, then (a, b, c) is completable.

1. INTRODUCTION

Let A be a commutative ring with identity. A row vector $\underline{a} = (a_1, a_2, \ldots, a_n)$ with $a_i \in A$ is called a *unimodular row* if there exists $b_i \in A$ such that $\sum a_i b_i = 1$. Thus given a unimodular row $\underline{a}$, we get an exact sequence $0 \to A \xrightarrow{\underline{a}} A^n \to P \to 0$, where P is a projective module over A of rank $n-1$ and $P \oplus A \cong A^n$. We call P the projective module associated to the unimodular row $\underline{a}$. So P is stably free. In general, it is important and interesting to find conditions on such a unimodular row so that the associated projective module is free. It is immediate that this is equivalent to the condition that there exists a non-singular matrix σ of size n with entries in A such that

$$(a_1, a_2, \ldots, a_n)\sigma = (1, 0, \ldots, 0).$$

Such a unimodular row is called *completable.*

The first non-trivial condition of this kind was obtained by Swan and Towber [4] and was later generalized by A. A. Suslin [3].

Theorem 1 (Suslin). *Let $(a_1, a_2, \ldots, a_n)$ be a unimodular row over a ring A. Assume that $a_i = b_i^{r_i}$ where r_is are non-negative integers and $b_i \in A$. If $(n-1)!$ divides $\prod r_i$, then $(a_1, a_2, \ldots, a_n)$ is completable.*

In an attempt to generalize the above theorem, M. V. Nori conjectured the following.

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Conjecture 1 (Nori)**.** Let $\phi : R = k[x_1, x_2, \ldots, x_n] \to A$ be a homomorphism of k-algebras where k is a field and the x_i's are indeterminates. Assume that the row $(\phi(x_1), \ldots, \phi(x_n))$ is unimodular. Let $f_i \in R$ with $1 \le i \le n$ be such that the radical of the ideal generated by the f_i's is $(x_1, \ldots, x_n)$ and the length of $R/(f_1, \ldots, f_n)$ is a multiple of $(n-1)!$. Then the unimodular row $(a_1, a_2, \ldots, a_n)$ where $a_i = \phi(f_i)$ is completable.

This conjecture is still open, but the first author proved a partial result in this direction [2].

Theorem 2 (Mohan Kumar)**.** *Assume in the above that k is algebraically closed and the f_i's are homogeneous. Then the conjecture is true.*

In the present article we prove yet another sufficient condition for a unimodular row of length three to be completable, which does not seem to follow from the above results.

Theorem 3. *Let $(a, b, c) \in A^3$ be unimodular. Suppose that the polynomial $z^2 + bz + ac$ has a root in A. Then (a, b, c) is completable.*

As an immediate corollary we have

Corollary 4. *Suppose $1/2 \in A$ and $(a, b, c) \in A^3$ unimodular. If $b^2 - 4ac$ is a square in A, then (a, b, c) is completable.*

For larger length unimodular rows we have a somewhat weaker result which can be found in section 3.

2. Proof of the theorem

We will give two proofs of the theorem. As always, A be will be a commutative ring with identity. In all the results, since only finitely many elements of the ring are involved we may assume that A is finitely generated over the prime ring. Thus, we may assume that A is Noetherian.

Lemma 5 (Swan)**.** *Let L be an r- generated line bundle over A. Then for any integer $n \ge 0$, L^n is r-generated. In particular, if $r = 2$, $L^n \oplus L^{-n}$ is free.*

Proof. Let $l_1, l_2, \ldots, l_r \in L$ generate L. Then it is immediate that $l_i^n \in L^n$ generate L^n, by a local checking. In particular, if $r = 2$, we have a surjection $A^2 \to L^n$ and by determinant considerations, the kernel of this map is L^{-n} and thus $L^n \oplus L^{-n} \cong A^2$. □

Let $\pi : \mathbb{P}^1_A \to \operatorname{Spec} A$ be the structure morphism and let $\mathcal{O}_{\mathbb{P}^1_A}(1)$ be the tautlogical line bundle.Then $\pi_*\mathcal{O}_{\mathbb{P}^1_A}(n) = \mathrm{H}^0(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(n))$ is a free A-module of rank $n+1$ for all $n \geq 0$. Let x, y be the homogeneous coordinates of $\mathbb{P}^1_A$. Given $(a, b, c) \in A^3$, unimodular, we may define a subscheme X of $\mathbb{P}^1_A$ by the vanishing of $s = ax^2 + bxy + cy^2$. Then we have an exact sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^1_A}(-2) \xrightarrow{s} \mathcal{O}_{\mathbb{P}^1_A} \to \mathcal{O}_X \to 0, \tag{1}$$

The unimodularity of (a, b, c) implies that the restriction map $\pi : X \to \operatorname{Spec} A$ is a projective morphism of degree two. In particular it is quasi-finite. Thus by [Chap. III, 11.2 and Chap. II, Exercises 5.17(b)] in Hartshorne's book [1], X is affine and so let us write $X = \operatorname{Spec} B$ where $B = \mathrm{H}^0(X, \mathcal{O}_X)$. Taking cohomology of the sequence (1), we get,

$$0 \to A = \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1_A}) \to B = \mathrm{H}^0(\mathcal{O}_X) \to \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^1_A}(-2)) \cong A \to 0.$$

So we see that $B = A \oplus A$ as A-modules.

We first identify the ring B. B is generated by one element over A. Since the generator is identified as a generator of $\mathrm{H}^1(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(-2))$, we use Čech complex to identify this element. Let U_x (resp. U_y) be the open set $x \neq 0$ (resp. $y \neq 0$) of $\mathbb{P}^1_A$. With respect to this open cover, the Čech complexes for the exact sequence (1) fits into a diagram as follows.

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{H}^0(U_x, \mathcal{O}_{\mathbb{P}^1_A}(-2)) \oplus \mathrm{H}^0(U_y, \mathcal{O}_{\mathbb{P}^1_A}(-2)) & \xrightarrow{\phi_1} & \mathrm{H}^0(U_x \cap U_y, \mathcal{O}_{\mathbb{P}^1_A}(-2)) \\
\psi_1 \downarrow & & \psi_2 \downarrow \\
\mathrm{H}^0(U_x, \mathcal{O}_{\mathbb{P}^1_A}) \oplus \mathrm{H}^0(U_y, \mathcal{O}_{\mathbb{P}^1_A}) & \xrightarrow{\phi_2} & \mathrm{H}^0(U_x \cap U_y, \mathcal{O}_{\mathbb{P}^1_A}) \\
\downarrow & & \downarrow \\
\mathrm{H}^0(U_x, \mathcal{O}_X) \oplus \mathrm{H}^0(U_y, \mathcal{O}_X) & \xrightarrow{\phi_3} & \mathrm{H}^0(U_x \cap U_y, \mathcal{O}_X)
\end{array}$$

We write these in terms of the corresponding rings. Denote by h the polynomial $at^2 + bt + c \in A[t]$, where $t = y/x$.

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ A[t] \oplus A[t^{-1}] & \stackrel{\phi_1}{\to} & A[t, t^{-1}] \\ \psi_1 \downarrow & & \psi_2 \downarrow \\ A[t] \oplus A[t^{-1}] & \stackrel{\phi_2}{\to} & A[t, t^{-1}] \\ \downarrow & & \downarrow \\ A[t]/(h) \oplus A[t^{-1}]/(ht^{-2}) & \stackrel{\phi_3}{\to} & A[t, t^{-1}]/(h) \end{array}$$

Next we explicitly describe the maps which appear above.

$$\begin{aligned} \phi_1(\alpha, \beta) &= \alpha - t^{-2}\beta \\ \phi_2(\alpha, \beta) &= \alpha - \beta \\ \phi_3(\alpha, \beta) &= \alpha - \beta \\ \psi_1(\alpha, \beta) &= (h\alpha, ht^{-2}\beta) \\ \psi_2(\alpha) &= h\alpha \end{aligned}$$

The following are easy to check. Kernel of ϕ_1 is zero (since it is $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1_A}(-2))$) and similarly, kernel of ϕ_2 is A and kernel of ϕ_3 is B. Cokernel of ϕ_1 is naturally identified with At^{-1} and cokernels of both ϕ_2, ϕ_3 are zero. The element $z = (at, -(b + ct^{-1})) \in A[t]/(h) \oplus A[t^{-1}]/(ht^{-2})$ goes to zero under ϕ_3 and hence defines an element in B. We claim that this element generates B as an A-algebra. To check this, suffices to check that this element goes to $t^{-1} \in At^{-1} = \mathrm{H}^1(\mathcal{O}_{\mathbb{P}^1_A}(-2))$. We do this by a simple diagram chase. Clearly z can be lifted to $(at, -(b + ct^{-1})) \in A[t] \oplus A[t^{-1}]$. When we apply ϕ_2 to this we get $ht^{-1} \in A[t, t^{-1}]$ and thus it is the image of t^{-1} under ψ_2 proving the claim. Next I claim that z satisfies the equation $z^2 + bz + ac = 0$. Suffices to check that $at \in A[t]/(h)$ and $-(b + ct^{-1}) \in A[t^{-1}]/(ht^{-2})$ satisfy this equation.

$$\begin{aligned} (at)^2 + b(at) + ac &= a(at^2 + bt + c) = ah = 0, \\ (-(b + ct^{-1}))^2 + b(-(b + ct^{-1})) + ac &= \\ &= b^2 + 2bct^{-1} + c^2t^{-2} - b^2 \\ &\quad - bct^{-1} + ac \\ &= bct^{-1} + c^2t^{-2} + ac \\ &= cht^{-2} = 0 \end{aligned}$$

Thus we have identified B to be $A[z]/(z^2 + bz + ac)$.

Since X is affine, for any sheaf $\mathcal{F}$ on X, the natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is surjective. Twisting the exact sequence (1) by $\mathcal{O}_{\mathbb{P}^1_A}(2)$ and taking cohomologies, we get

$$0 \to A \xrightarrow{(a,b,c)} A^3 \to P = \pi_* \mathcal{O}_X(2) \to 0 \tag{2}$$

We have a surjection $\pi^*(P) \to \mathcal{O}_X(2)$ and thus we see that $\pi^*(P) = \mathcal{O}_X(2) \oplus \mathcal{O}_X(-2)$. But we have seen that $\pi_* \mathcal{O}_X(1) = \pi_* \mathcal{O}_{\mathbb{P}^1_A}(1) = A^2$ and thus $\mathcal{O}_X(1)$ is two generated. So, by lemma 5, we see that $\pi^* P$ is free. If $z^2 + bz + ac$ has a root in A, we have a retraction $B \to A$. Since $B \otimes_A P$ is free, we see that P must be free as well, proving the theorem.

Remark 1. The polynomial $z^2+bz+ac = 0$ has a root in A is equivalent to saying that the map $\pi : X \to \operatorname{Spec} A$ has a section.

Remark 2 (Nori). Let $(a, b, c) \in A^3$ be unimodular and P the associated projective module. Then for any $n \geq 2$, there exists an A-algebra B, which is A-free of rank n and $B \otimes_A P$ is B-free.

Proof. Let $s = ax^n + bx^{n-1}y + cy^n$, where as before, x and y are the homogeneous coordinates of $\mathbb{P}^1_A$. The zeroes of s define a subscheme X of $\mathbb{P}^1_A$. We have an exact sequence,

$$0 \to \mathcal{O}_{\mathbb{P}^1_A}(-n) \xrightarrow{s} \mathcal{O}_{\mathbb{P}^1_A} \to \mathcal{O}_X \to 0.$$

Then as before, $\mathrm{H}^0(X, \mathcal{O}_X) = B$ is a free A-module of rank n, since $\mathrm{H}^1(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(-n))$ is a free A-module of rank $n-1$ and the map $X \to \operatorname{Spec} A$ is a finite map of degree n since (a, b, c) is unimodular. We have as before, a surjection $B^2 \to \mathcal{O}_X(n)$ given by the basis elements going to x^n, y^n. Thus we get a surjection $B^3 \to \mathcal{O}_X(n)$, by sending the basis elements to $x^n, x^{n-1}y, y^n$. Since $ax^n + bx^{n-1}y + cy^n = 0$ on X, we see that this surjection factors through $B^3/B(a, b, c) = Q$. Thus the B-projective module Q is isomorphic to $\mathcal{O}_X(n) \oplus \mathcal{O}_X(-n)$ and since $\mathcal{O}_X(n)$ is two generated, by lemma 5, we see that Q is free. But,

$$Q = B^3/B(a, b, c) = A^3/A(a, b, c) \otimes_A B = P \otimes_A B.$$

□

Next we give another proof of the theorem. Consider the ring

$$B = \mathbb{Z}[a, b, c, a', b', c', z]/(aa' + bb' + cc' - 1, z^2 + bz + ac),$$

where a, b, c, a', b', c', z are indeterminates. Then the ideal $I = (z, a)$ is locally free. This is easily seen as follows. Let $\mathfrak{M}$ be any maximal ideal of B. If either $z \notin \mathfrak{M}$ or $a \notin \mathfrak{M}$, clearly I is locally free at $\mathfrak{M}$. Since $z(z+b)+ac = 0$, again, if $z+b \notin \mathfrak{M}$ or $c \notin \mathfrak{M}$, I is locally free. Thus we may assume that $\mathfrak{M}$ contains $z, z+b, a, c$ which implies $a, b, c \in \mathfrak{M}$ and there are no such maximal ideals, since (a, b, c) is unimodular. Then as in lemma 5, we see that $I^2 = (z^2, az, a^2)$ is in fact generated by (z^2, a^2). We have a surjective map $\phi : B^3 \to I^2$ given by

$$\phi(e_1) = z^2, \phi(e_2) = az, \phi(e_3) = a^2$$

where the e_i's form a basis for B^3. But,

$$\phi(ae_1 + be_2 + ce_3) = az^2 + abz + a^2c = a(z^2 + bz + ac) = 0.$$

Thus ϕ factors through $P = B^3/(ae_1 + be_2 + ce_3)$. So, we see that $P \cong I^2 \oplus I^{-2}$ and since I^2 is two generated, P is free by lemma 5. Now, if A is a ring with a unimodular row which also we call by abuse of notation a, b, c and such that the polynomial $z^2 + bz + ac$ has a root in A, then writing $aa' + bb' + cc' = 1$ for suitable $a', b', c' \in A$, we get a homomorphism $\psi : B \to A$ in the obvious fashion, sending z to a root of the polynomial $z^2 + bz + ac$, which we have assumed exists in A. Then the associated projective module over A is just $P \otimes_B A$ and since we have seen that P is free, we are done.

3. Unimodular rows of length more than three

In this section we consider unimodular rows $(a_1, a_2, \ldots, a_n)$ with n at least three. Notation will be as before. We will also assume that A satisfies the following condition on A which was not necessary for $n = 3$ case.

(*) *Any two generated line bundle on A is free. For example A is a UFD.*

As before, given $(a_1, a_2, \ldots, a_n) \in A^n$ a unimodular row, we consider the polynomial $s(x, y) = a_1x^{n-1} + a_2x^{n-2}y + \cdots + a_ny^{n-1}$ and define $X \subset \mathbb{P}^1_A$ to be the subscheme defined by the vanishing of s. Then, as before the map $\pi : X \to \operatorname{Spec} A$ is a finite map of degree $n - 1$ and in particular X is affine, since we may assume that A is Noetherian.

Theorem 6. *With the notation as in the previous paragraph, if $\pi : X \to \operatorname{Spec} A$ has a section then the unimodular row is completable.*

Proof. Let $\epsilon : \operatorname{Spec} A \to X$ be a section. Then we can consider it as a section $\epsilon : \operatorname{Spec} A \to \mathbb{P}^1_A$ whose image is contained in X. But any such section is given by a surjection from $V = A^2 = \mathrm{H}^0(\mathbb{P}^1_A, \mathcal{O}_{\mathbb{P}^1_A}(1))$ to a line bundle (which is just $\epsilon^*(\mathcal{O}_{\mathbb{P}^1_A}(1))$) on A. By our hypothesis, this line bundle is trivial. So, this is just choosing a basis for V. In other words, giving such a section is the same as a change of variables $(x, y) \mapsto (u, v)$ and then the section is contained in X implies that the equation s after a change of variables is divisible by say u. On the other hand, the change of variables which is just an A-automorphism of V induces an automorphism $S^{n-1}V \to S^{n-1}V$ and s maps to an s' divisible by u.

Thus in the new variables, we get $s' = a'_1 u^n + \cdots + a'_{n-1} uv^{n-1}$ and since the quotient of $S^{n-1}V$ by s or s' gives the same projective module, we see that $(a_1, a_2, \ldots, a_n) \sim (a'_1, \ldots, a'_{n-1}, 0)$ and thus completable. $\square$

Remark 3. From the proof above, we see that with the condition (*) on A, having a section of $\pi : X \to \operatorname{Spec} A$ implies that s has a linear factor. Conversely, if s has a linear factor, say $bx - ay$, then the coefficients of s are contained in the ideal (a, b) and thus (a, b) is unimodular, since the coefficients of s generate the unit ideal. Then it is clear that the map $\pi : X \to \operatorname{Spec} A$ has a section. Thus the above result can be restated as follows. The unimodular row $(a_1, a_2, \ldots, a_n)$ is completable if the associated polynomial s has a linear factor.

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