FOUR-BY-FOUR PFAFFIANS

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This paper is dedicated to Paolo Valabrega on his sixtieth birthday.

Abstract. This paper shows that the general hypersurface of degree ≥ 6 in projective four space cannot support an indecomposable rank two vector bundle which is Arithmetically Cohen-Macaulay and four generated. Equivalently, the equation of the hypersurface is not the Pfaffian of a four by four minimal skew-symmetric matrix.

1. Introduction

In this note, we study indecomposable rank two bundles E on a smooth hypersurface X in $\mathbf{P}^4$ which are Arithmetically Cohen-Macaulay. The existence of such a bundle on X is equivalent to X being the Pfaffian of a minimal skew-symmetric matrix of size $2k \times 2k$, with $k \geq 2$. The general hypersurface of degree ≤ 5 in $\mathbf{P}^4$ is known to be Pfaffian ([1], [2] [5]) and the general sextic in $\mathbf{P}^4$ is known to be not Pfaffian ([4]). One should expect the result of [4] to extend to all general hypersurfaces of degree ≥ 6. (Indeed the analogous statement for hypersurfaces in $\mathbf{P}^5$ was established in [7].) However, in this note we offer a partial result towards that conclusion. We show that the general hypersurface in $\mathbf{P}^4$ of degree ≥ 6 is not the Pfaffian of a 4×4 skew-symmetric matrix. For a hypersurface of degree r to be the Pfaffian of a $2k \times 2k$ skew-symmetric matrix, we must have $2 \leq k \leq r$. It is quite easy to show by a dimension count that the general hypersurface of degree $r \geq 6$ in $\mathbf{P}^4$ is not the Pfaffian of a $2r \times 2r$ skew-symmetric matrix of linear forms. Thus, this note addresses the lower extreme of the range for k.

2. Reductions

Let X be a smooth hypersurface on $\mathbf{P}^4$ of degree $r \geq 2$. A rank two vector bundle E on X will be called Arithmetically Cohen-Macaulay (or ACM) if $\oplus_{k\in\mathbb{Z}} H^i(X, E(k))$ equals 0 for $i = 1, 2$. Since $\mathrm{Pic}(X)$ equals $\mathbb{Z}$, with generator $\mathcal{O}_X(1)$, the first Chern class $c_1(E)$ can be treated as an integer t. The bundle E has a minimal resolution over $\mathbf{P}^4$ of the form

$$0 \to L_1 \xrightarrow{\phi} L_0 \to E \to 0,$$

where L_0, L_1 are sums of line bundles. By using the isomorphism of E and $E^\vee(t)$, we obtain (see [2]) that $L_1 \cong L_0^\vee(t-r)$ and the matrix ϕ (of homogeneous polynomials) can be chosen as skew-symmetric. In particular, F_0 has even rank and the defining polynomial of X is the Pfaffian of this matrix. The case where ϕ is two by two is just the case where E is decomposable. The next case is where ϕ is a four by four minimal matrix. These correspond to ACM bundles E with four global sections (in possibly different degrees) which generate it.

Our goal is to show that the generic hypersurface of degree $r \geq 6$ in $\mathbf{P}^4$ does not support an indecomposable rank two ACM bundle which is four generated, or equivalently, that such a hypersurface does not have the Pfaffian of a four by four minimal matrix as its defining polynomial.

So fix a degree $r \geq 6$. Let us assume that E is a rank two ACM bundle which is four generated and which has been normalized so that its first Chern class t equals 0 or -1. If $L_0 = \oplus_{i=1}^4 \mathcal{O}_{\mathbf{P}}(a_i)$ with $a_1 \geq a_2 \geq a_3 \geq a_4$, the resolution for E is given by

$$\oplus_{i=1}^4 \mathcal{O}_{\mathbf{P}}(t-a_i-r) \xrightarrow{\phi} \oplus_{i=1}^4 \mathcal{O}_{\mathbf{P}}(a_i).$$

Write the matrix of ϕ as

$$\phi = \begin{bmatrix} 0 & A & B & C \\ -A & 0 & D & E \\ -B & -D & 0 & F \\ -C & -E & -F & 0 \end{bmatrix}.$$

Since X is smooth with equation $AF - BE + CD = 0$, the homogeneous entries A, B, C, D, E, F are all non-zero and have no common zero on $\mathbf{P}^4$.

Lemma 2.1. *For fixed r and t (normalized), there are only finitely many possibilities for (a_1, a_2, a_3, a_4).*

Proof. Let a, b, c, d, e, f denote the degrees of the poynomials A, B, C, D, E, F. Since the Pfaffian of the matrix is $AF - BE + CD$, the degree of each matrix entry is bounded between 1 and $r-1$. $a = a_1 + a_2 + (r-t), b = a_1 + a_3 + (r-t)$ etc. Thus if $i \neq j$, $0 < a_i + a_j + r - t < r$ while $\sum a_i = -r + 2t$. From the inequality, regardless of the sign of a_1, the other three values a_2, a_3, a_4 are < 0. But again using the inequality, their pairwise sums are $> -r + t$, hence there are only finitely many choices for them. Lastly, a_1 depends on the remaining quantities. ♣

It suffices therefore to fix $r \geq 6$, $t = 0$ or -1 and a four-tuple (a_1, a_2, a_3, a_4) and show that there is no ACM bundle on the general hypersurface of degree r which has a resolution given by a matrix ϕ of the type $(a_1, a_2, a_3, a_4), t$.

¿From the inequalities on a_i, we obtain the inequalities

$$0 < a \leq b \leq c, d \leq e \leq f < r.$$

We do no harm by rewriting the matrix ϕ with the letters C and D interchanged to assume without loss of generality that $c \leq d$.

Proposition 2.2. *Let X be a smooth hypersurface of degree ≥ 3 in $\mathbf{P}^4$ supporting an ACM bundle E of type $(a_1 \geq a_2 \geq a_3 \geq a_4), t$. The degrees of the entries of ϕ can be arranged (without loss of generality) as:*

$$a \leq b \leq c \leq d \leq e \leq f$$

Then X will contain a curve Y which is the complete intersection of hypersurfaces of the three lowest degrees in the arrangement and a curve Z which is the complete intersection of hypersurfaces of the three highest degrees in the arrangement.

Proof. Consider the ideals (A, B, C) and $D, E, F)$. Since the equation of X is $AF - BE + CD$, these ideals give subschemes of X. Take for example (A, B, C). If the variety Y it defines has a surface component, this gives a divisor on X. As Pic(X)= $\mathbb{Z}$, there is a hypersurface $S = 0$ in $\mathbf{P}^4$ inducing this divisor. Now at a point in $\mathbf{P}^4$ where $S = D = E = F = 0$, all six polynomials $A, \ldots, F$ vanish, making a multiple point for X. Hence, X being smooth, Y must be a curve on X. Thus (A, B, C) defines a complete intersection curve on X. ♣

To make our notations non-vacuous, we will assume that at least one smooth hypersurface exists of a fixed degree $r \geq 6$ with an ACM bundle of type $(a_1 \geq a_2 \geq a_3 \geq a_4), t$. Let $\mathcal{F}_{(a,b,c);r}$ denote the Hilbert flag scheme that parametrizes all inclusions $Y \subset X \subset \mathbf{P}^4$ where X is a hypersurface of degree r and Y is a complete intersection curve lying on X which is cut out by three hypersurfaces of degrees a, b, c. Our discussion above produces points in $\mathcal{F}_{(a,b,c);r}$ and $\mathcal{F}_{(d,e,f);r}$.

Let $\mathcal{H}_r$ denote the Hilbert scheme of all hypersurfaces in $\mathbf{P}^4$ of degree r and let $\mathcal{H}_{a,b,c}$ denote the Hilbert scheme of all curves in $\mathbf{P}^4$ with the same Hilbert polynomial as the complete intersection of three hypersurfaces of degrees a, b and c. Following J. Kleppe ([6]), the Zariski tangent spaces of these three schemes are related as follows: Corresponding to the projections

$$\begin{array}{ccc} \mathcal{F}_{(a,b,c);r} & \xrightarrow{p_2} & \mathcal{H}_{a,b,c} \\ \downarrow p_1 & & \\ \mathcal{H}_r & & \end{array}$$

if T is the tangent space at the point $Y \overset{i}{\subset} X \subset \mathbf{P}^4$ of $\mathcal{F}_{(a,b,c);r}$, there is a Cartesian diagram

$$\begin{array}{ccc} T & \xrightarrow{p_2} & H^0(Y, \mathcal{N}_{Y/\mathbf{P}}) \\ \downarrow p_1 & & \downarrow \alpha \\ H^0(X, \mathcal{N}_{X/\mathbf{P}}) & \xrightarrow{\beta} & H^0(Y, i^*\mathcal{N}_{X/\mathbf{P}}) \end{array}$$

of vector spaces.

Hence $p_1 : T \to H^0(X, \mathcal{N}_{X/\mathbf{P}})$ is onto if and only if $\alpha : H^0(Y, \mathcal{N}_{Y/\mathbf{P}}) \to H^0(Y, i^*\mathcal{N}_{X/\mathbf{P}})$ is onto. The map α is easy to describe. It is the map given as

$$H^0(Y, \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \xrightarrow{[F,-E,D]} H^0(Y, \mathcal{O}_Y(r)).$$

Hence

Proposition 2.3. *Choose general forms A, B, C, D, E, F of degrees a, b, c, d, e, f and let Y be the curve defined by $A = B = C = 0$. If the map*

$$H^0(Y, \mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \xrightarrow{[F,-E,D]} H^0(Y, \mathcal{O}_Y(r))$$

is not onto, then the general hypersurface of degree r does not support a rank two ACM bundle of type $(a_1, a_2, a_3, a_4), t$.

Proof. Consider a general Pfaffian hypersurface X of equation $AF - BE + CD = 0$ where A, B, C, D, E, F are chosen generally. Such an X contains such a Y and X is in the image of p_1. By our hypothesis, $p_1 : T \to H^0(X, \mathcal{N}_{Y/\mathbf{P}})$ is not onto and (in characteristic zero) it follows that $p_1 : \mathcal{F}_{(a,b,c);r} \to \mathcal{H}_r$ is not dominant. Since all hypersurfaces X supporting such a rank two ACM bundle are in the image of p_1, we are done. ♣

Remark 2.4. Note that the last proposition can also be applied to the situation where Y is replaced by the curve Z given by $D = E = F = 0$, with the map given by $[A, -B, C]$, with a similar statement.

3. Calculations

We are given general forms A, B, C, D, E, F of degrees a, b, c, d, e, f where $a+f = b+e = c+d = r$ and where without loss of generality, by interchanging C and D we may assume that $1 \leq a \leq b \leq c \leq d \leq e \leq f < r$. Assume that $r \geq 6$. We will show that if Y is the curve $A = B = C = 0$ or if Z is the curve $D = E = F = 0$,depending on the conditions on a, b, c, d, e, f, either $h^0(\mathcal{N}_{Y/\mathbf{P}}) \overset{[F,-E,D]}{\longrightarrow} h^0(\mathcal{O}_Y(r))$ or $h^0(\mathcal{N}_{Z/\mathbf{P}}) \overset{[A,-B,C]}{\longrightarrow} h^0(\mathcal{O}_Z(r))$ is not onto. This will prove the desired result.

3.1. **Case 1.** $b \geq 3, c \geq a+1, 2a+b < r-2$.
In $\mathbf{P}^5$ (or in 6 variables) consider the homogeneous complete intersection ideal

$$I = (X_0^a, X_1^b, X_2^c, X_3^{r-c}, X_4^{r-b}, X_5^{r-a} - X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+2})$$

in the the polynomial ring S_5 on $X_0, \ldots, X_5$. Viewed as a module over S_4 (the polynomial ring on $X_0, \ldots, X_4$), $M = S_5/I$ decomposes as a direct sum

$$M = N(0) \oplus N(1)X_5 \oplus N(2)X_5^2 \oplus \cdots \oplus N(r-a-1)X_5^{r-a-1},$$

where the $N(i)$ are graded S_4 modules. Consider the multiplication map $X_5 : M_{r-1} \to M_r$ from the $(r-1)$-st to the r-th graded pieces of M. We claim it is injective and not surjective.

Indeed, any element m in the kernel is of the form nX_5^{r-a-1} where n is a homogeneous element in $N(r-a-1)$ of degree a. Since $X_5 \cdot m = n \cdot X_5^{r-a} \equiv n \cdot X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+2} \equiv 0 \mod (X_0^a, X_1^b, X_2^c, X_3^{r-c}, X_4^{r-b})$ we may assume that n itself is represented by a monomial in $X_0, \ldots, X_4$ of degree a. Our inequalities have been chosen so that even in the case where n is represented by X_4^a, the exponents of X_4 in the product is $a+a+2$ which is less than $r-b$. Thus n and hence the kernel must be 0.

On the other hand, the element $X_0^{a-1}X_1^2X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+1}$ in M_r lies in its first summand $N(0)_r$. In order to be in the image of multiplication by X_5, this element must be a multiple of $X_2^{c-a-1}X_3^{r-c-a-1}X_4^{a+2}$. By inspecting the factor in X_4, this is clearly not the case. So the multiplication map is not surjective.

Hence dim $M_{r-1} <$ dim M_r. Now the Hilbert function of a complete intersection ideal like I depends only on the degrees of the generators. Hence, for any complete intersection ideal I' in S_5 with generators of the same degrees, for the corresponding module $M' = S_5/I'$, dim M'_{r-1} $<$ dim M'_r.

Now coming back to our general six forms A, B, C, D, E, F in S_4, of the same degrees as the generators of the ideal I above. Since they include a regular sequence on $\mathbf{P}^4$, we can lift these polynomials to forms A', B', C', D', E', F' in S_5 which give a complete intersection ideal I' in S_5.

The module $\bar{M} = S_4/(A, B, C, D, E, F)$ is the cokernel of the map

$$X_5 : M'(-1) \to M'.$$

By our argument above, we conclude that $\bar{M}_r \neq 0$.

Lastly, the map $H^0(\mathcal{O}_Y(a) \oplus \mathcal{O}_Y(b) \oplus \mathcal{O}_Y(c)) \stackrel{[F,-E,D]}{\to} H^0(\mathcal{O}_Y(r))$ has cokernel precisely $\bar{M}_r$ which is not zero, and hence the map is not onto.

3.2. **Case 2.** $b \leq 2$

Since the forms are general, the curve Y given by $A = B = C = 0$ is a smooth complete intersection curve, with $\omega_Y \cong \mathcal{O}_Y(a + b + c - 5)$. Since $a + b \leq 4$, $\mathcal{O}_Y(c)$ is nonspecial

(1) Suppose $\mathcal{O}_Y(a)$ is nonspecial. Then all three of $\mathcal{O}_Y(a), \mathcal{O}_Y(b), \mathcal{O}_Y(c)$ are nonspecial. Hence $h^0(\mathcal{N}_{Y/\mathbf{P}}) = (a + b + c)\delta + 3(1 - g)$ where $\delta = abc$ is the degree of Y and g is the genus. Also $h^0(\mathcal{O}_Y(r)) = r\delta + 1 - g + h^1(\mathcal{O}_Y(r)) \geq r\delta + 1 - g$. To show that $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$, it is enough to show that

$$(a + b + c)\delta + 3(1 - g) < r\delta + 1 - g.$$

Snce $2g - 2 = (a + b + c - 5)\delta$, this inequality becomes $5\delta < r\delta$ which is true as $r \geq 6$.

(2) Suppose $\mathcal{O}_Y(a)$ is special (so $b + c \geq 5$), but $\mathcal{O}_Y(b)$ is nonspecial. By Cliffords theorem, $h^0(\mathcal{O}_Y(a)) \leq \frac{1}{2}a\delta + 1$. In this case $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided that
$\frac{1}{2}a\delta + 1 + (b + c)\delta + 2(1 - g) < r\delta + (1 - g)$
or $r > \frac{b+c}{2} + \frac{1}{\delta} + \frac{5}{2}$.
Since $c \leq \frac{r}{2}$ and $b \leq 2$, this is achieved if
$r > \frac{2+r/2}{2} + \frac{1}{\delta} + \frac{5}{2}$ which is the same as $r > \frac{14}{3} + \frac{4}{3\delta}$.
But $c \geq 3$, so $\delta \geq 3$, hence the last inequality is true as $r \geq 6$.

(3) Suppose both $\mathcal{O}_Y(a)$ and $\mathcal{O}_Y(b)$ are special. Hence $a + c \geq 5$. Using Cliffords theorem, in this case $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided that
$\frac{1}{2}(a + b)\delta + 2 + c\delta + (1 - g) < r\delta + (1 - g)$.
This becomes $r > \frac{1}{2}(a + b) + \frac{2}{\delta} + c$. Using $c \leq \frac{r}{2}$, $a + b \leq 4$, and $\delta \geq 3$, this is again true when $r \geq 6$.

3.3. **Case 3.** $c < a+1$.
In this case $a = b = c$ and $r \geq 2a$. Using the sequence

$$0 \to \mathcal{I}_Y(a) \to \mathcal{O}_{\mathbf{P}}(a) \to \mathcal{O}_Y(a) \to 0,$$

we get $h^0(\mathcal{N}_{Y/\mathbf{P}}) = 3h^0(\mathcal{O}_Y(a)) = 3[\binom{a+4}{4} - 3]$
while $h^0(\mathcal{O}_Y(r)) \geq h^0(\mathcal{O}_Y(2a)) = \binom{2a+4}{4} - 3\binom{a+4}{4} + 3$. Hence the inequality $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ will be true provided
$\binom{2a+4}{4} > 6\binom{a+4}{4} - 12$.
The reader may verify that is reduces to
$10a^4 + 20a^3 - 70a^2 - 200a + 7(4!) > 0$
and the last inequality is true when $a \geq 3$. Thus we have settled this case when $r \geq 6$ and $a \geq 3$. If $r \geq 6$ and a (and hence b) ≤ 2, we are back in the previous case.

3.4. **Case 4.** $2a + b \geq r - 2$ and $r \geq 82$.
For this case, we will study the curve Z given by $D = E = F = 0$ (of degrees $r - c, r - b, r - a$) and consider the inequality $h^0(\mathcal{N}_{Z/\mathbf{P}}) < h^0(\mathcal{O}_Z(r))$
Since $a, b, c \leq \frac{r}{2}$, $2a + 2 \geq r - b \geq \frac{r}{2}$, hence $a \geq \frac{r}{4} - 1$. Also $b \geq a$ and $2a + b \geq r - 2$, hence $b \geq \frac{r}{3} - \frac{2}{3}$. Likewise, $c \geq \frac{r}{3} - \frac{2}{3}$.

Now $h^0(\mathcal{O}_Z(r-a)) = h^0(\mathcal{O}_{\mathbf{P}}(r-a)) - h^0(\mathcal{I}_Z(r-a)) \leq \binom{r-a+4}{4} - 1$ etc., hence
$h^0(\mathcal{N}_{Z/\mathbf{P}}) \leq \binom{r-a+4}{4} + \binom{r-b+4}{4} + \binom{r-c+4}{4} - 3 \leq \binom{\frac{3r}{4}+5}{4} + 2\binom{\frac{2r}{3}+\frac{14}{3}}{4} - 3$
or $h^0(\mathcal{N}_{Z/\mathbf{P}}) \leq G(r)$, where $G(r)$ is the last expression.

Looking at the Koszul resolution for $\mathcal{O}_Z(r)$, since $a + b + c \leq \frac{3r}{2} < 2r$, the last term in the resolution has no global sections. Hence $h^0(\mathcal{O}_Z(r)) \geq h^0(\mathcal{O}_{\mathbf{P}}(r)) - [h^0(\mathcal{O}_{\mathbf{P}}(a)) + h^0(\mathcal{O}_{\mathbf{P}}(b)) + h^0(\mathcal{O}_{\mathbf{P}}(c))] \geq \binom{r+4}{4} - \binom{a+4}{4} - \binom{b+4}{4} - \binom{c+4}{4} \geq \binom{r+4}{4} - 3\binom{\frac{r}{2}+4}{4}$, or $h^0(\mathcal{O}_Z(r)) \geq F(r)$, where $F(r)$ is the last expression.

The reader may verify that $G(r) < F(r)$ for $r \geq 82$.

3.5. **Case 5.** $6 \leq r \leq 81$, $2a + b \geq r - 2$, $b \geq 3, c \geq a + 1$.
We still have $\frac{r}{4} - 1 \leq a \leq \frac{r}{2}, \frac{r}{3} - \frac{2}{3} \leq b, c \leq \frac{r}{2}$. For the curve Y given by $A = B = C = 0$, we can explicitly compute $h^0(\mathcal{O}_Y(k))$ for any k using the Koszul resolution for $\mathcal{O}_Y(k)$. Hence both terms in the inequality $h^0(\mathcal{N}_{Y/\mathbf{P}}) < h^0(\mathcal{O}_Y(r))$ can be computed for all allowable values of a, b, c, r using a computer program like Maple and the inequality can be verified. We will leave it to the reader to verify this claim.

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