ON THE GEOMETRY OF GENERALISED QUADRICS

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1. Introduction

We work over the field of complex numbers $\mathbb{C}$.

Let $\{f_0, \cdots, f_n; g_0, \cdots, g_n\}$ be a regular sequence in $\mathbb{P}^{2n+1}$ and suppose that $Q = \sum_{i=0}^{n} f_i g_i$ is a homogeneous polynomial. We shall refer to the hypersurface X defined by Q as a *generalised quadric*. In this note, we prove that generalised quadrics in $\mathbb{P}^{2n+1}$ for $n \geq 1$ are reduced.

In characteristic $p > 0$, it is easy to construct generalised quadrics which are non-reduced. By exploiting this fact, low rank vector bundles were constructed on $\mathbb{P}^4$ and $\mathbb{P}^5$ in [4]. Furthermore, in characteristic 0, reducible generalised quadrics exist in $\mathbb{P}^3$; for instance, the hypersurface defined by $X^2Y^2 - Z^2U^2 = 0$, where X, Y, Z, U are the coordinates of $\mathbb{P}^3$, is such a generalised quadric. We do not know any examples of reducible generalised quadrics in higher dimensional projective spaces. However, the question of non-reducedness is settled by our main theorem.

2. Atiyah Class and Chern classes of vector bundles over schemes

Let X be any scheme and E be any vector bundle on X. We recall that the *Atiyah class* $\mathrm{at}(E)$ (see[1]) of the vector bundle E is the natural extension class

$$0 \to \Omega^1_X \otimes E \to \mathcal{P}(E) \to E \to 0$$

where $\mathcal{P}(E)$ is the *principal parts bundle* of E. Thus $\mathrm{at}(E)$ is an element of the cohomology group $\mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))$. Starting with this class, one can define *Chern-Hodge classes* $\mathrm{c}_i(E) \in \mathrm{H}^i(X, \Omega^i_X)$ as follows (see [3] or for a simpler exposition see [5]).

Consider the composition

$$(\Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m} \to (\Omega^1_X)^{\otimes m} \otimes \mathcal{E}nd(E^{\otimes m}) \to \Omega^m_X \otimes \mathcal{E}nd(\overset{\mathrm{m}}{\wedge} E) \to \Omega^m_X$$

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where the last map is induced by the trace map $\mathcal{E}nd(\overset{m}{\wedge} E) \to \mathcal{O}_X$. We then define (upto a constant non-zero factor) the Chern-Hodge classes from the composite map below:

$$\begin{array}{ccccc} \mathrm{H}^1(X, \Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m} & \to & \mathrm{H}^m(X, (\Omega^1_X \otimes \mathcal{E}nd(E))^{\otimes m}) & \to & \mathrm{H}^m(X, \Omega^m_X) \\ \mathrm{at}(E)^{\otimes m} & \mapsto & \mathrm{at}(E) \cup \cdots \cup \mathrm{at}(E) & \mapsto & \mathrm{c}_m(E) \end{array}$$

By convention, $c_0(E) = 1 \in \mathrm{H}^0(X, \mathcal{O}_X)$. Furthermore, we let $\mathrm{c}(E) = \sum \mathrm{c}_i(E)$ which is an invertible element in the graded commutative ring $\oplus_i \mathrm{H}^i(X, \Omega^i_X)$.

Now let X be any scheme and let $\mathcal{F}$ be a coherent sheaf on X which has a finite resolution by vector bundles

$$0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$$

Definition 1. $\mathrm{c}(\mathcal{F}) = \mathrm{c}(\mathrm{P}^\bullet_X) := \Pi_k \, \mathrm{c}(\mathrm{P}^k_X)^{(-1)^k} \in \oplus \mathrm{H}^i(X, \Omega^i_X)$.

We recall some basic properties of the Chern-Hodge classes. Let $\mathcal{P}(X)$ be the set of all sheaves on X which have a finite resolution by vector bundles.

Properties:

(1) For any sheaf $\mathcal{F} \in \mathcal{P}(X)$, $\mathrm{c}(\mathcal{F})$ is independent of the resolution.

(2) For any short exact sequence of sheaves in $\mathcal{P}(X)$

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

$\mathrm{c}(\mathcal{F}) = \mathrm{c}(\mathcal{F}')\,\mathrm{c}(\mathcal{F}'')$.

(3) For any morphism $f : Y \to X$, there is a natural ring homomorphism $f^* : \oplus_i \mathrm{H}^i(X, \Omega^i_X) \to \oplus_i \mathrm{H}^i(Y, \Omega^i_Y)$ under which if E is a bundle on X, then $f^* \mathrm{c}(E) = \mathrm{c}(f^* E)$.

(4) For any bundle E and a line bundle $\mathcal{L}$, we have

$$\mathrm{c}_r(E \otimes \mathcal{L}) = \sum_{i=0}^{r} \mathrm{c}_i(E)\,\mathrm{c}_1(\mathcal{L}^{r-i})$$

(5) If $\mathcal{F} \in \mathcal{P}(X)$ and

$$0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$$

is a finite resolution by vector bundles and if $f : Y \to X$ is any morphism, we can define $\mathrm{c}^Y(\mathcal{F}) \in \mathrm{H}^\bullet(Y, \Omega^\bullet_Y)$ as $\mathrm{c}(f^* \mathrm{P}^\bullet_X)$. In general, this is not $\mathrm{c}(f^*\mathcal{F})$, since this sheaf may not have a finite resolution by vector bundles on Y. These coincide if

$$0 \to f^* \mathrm{P}^\bullet_X \to f^*\mathcal{F} \to 0$$

remains exact and thus in this case $\mathrm{c}^Y(\mathcal{F}) = \mathrm{c}(f^*\mathcal{F})$.

(6) For any short exact sequence of sheaves in $\mathcal{P}(X)$

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

on X and a morphism $f : Y \to X$, $\mathrm{c}^Y(\mathcal{F}) = \mathrm{c}^Y(\mathcal{F}')\,\mathrm{c}^Y(\mathcal{F}'')$.

The following lemma, which is the key lemma, is essentially due to Gruson et.al [2].

Lemma 1. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Consider the restriction maps*

$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\alpha} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}})$$

and

$$\mathrm{H}^i(\Omega^i_X) \xrightarrow{\beta} \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}}).$$

Then $\mathrm{Im}\,\beta = \mathrm{Im}\,\alpha$ *for* $1 \le i < n-1$.

Proof. Since α factors through $\mathrm{H}^i(\Omega^i_X)$, we only need to show that $\mathrm{Im}\beta \subset \mathrm{Im}\alpha$. Since X is irreducible, we may assume that X is defined by a homogeneous polynomial $f^m, m > 1$ with f irreducible and so X_{red} is given by the vanishing of f. We consider the exact sequence

$$\mathcal{O}_X(-\deg(f^m)) \xrightarrow{d(f^m)} \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

Restricting it to X_{red}, we get

(1) $$\Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \cong \Omega^1_X \otimes \mathcal{O}_{X_{\mathrm{red}}}$$

This implies similar isomorphisms,

$$\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \cong \Omega^i_X \otimes \mathcal{O}_{X_{\mathrm{red}}},$$

for all i.

Since α factors through $\mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}})$ and similarly β factors through $\mathrm{H}^i(\Omega^i_X \otimes \mathcal{O}_{X_{\mathrm{red}}})$, it suffices to prove that the map

$$\mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \xrightarrow{\delta} \mathrm{H}^i(\Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}})$$

is onto by the isomorphism (1) above. We have an exact sequence,

$$0 \to \Omega^i_{\mathbb{P}^n}(-d) \to \Omega^i_{\mathbb{P}^n} \to \Omega^i_{\mathbb{P}^n} \otimes \mathcal{O}_{X_{\mathrm{red}}} \to 0,$$

where $d = \deg f$. Taking cohomologies and noting that $\mathrm{H}^j(\Omega^i_{\mathbb{P}^n}(-d)) = 0$ for $j = i, i+1$, since $1 \le i < n-1$, we see that δ is an isomorphism. $\square$

Lemma 2. *Let $M \subset \mathbb{P}^n$ be a closed subscheme of dimension r. Then the natural map,*

$$\gamma : \mathrm{H}^i(\Omega^i_{\mathbb{P}^n}) \to \mathrm{H}^i(\Omega^i_M)$$

is injective for $0 \le i \le r$.

Proof. If $h \in \mathrm{H}^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$ is the class of the hyperplane section, then $\mathrm{H}^i(\mathbb{P}^n, \Omega^i_{\mathbb{P}^n})$ is a one dimensional vector space generated by h^i. Thus, it suffices to show that its image in $\mathrm{H}^i(M, \Omega^i_M)$ is non-zero. If it is zero for some $i < r$, then $h^r = h^i h^{r-i} = 0 \in \mathrm{H}^r(M, \Omega^r_M)$. A proof of the well known fact that $h^r \neq 0$ is sketched in the Appendix. □

Lemma 3. *Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface which is not reduced. Let $\mathcal{F}$ be a coherent sheaf on X with a resolution $0 \to \mathrm{P}^\bullet_X \to \mathcal{F} \to 0$ by vector bundles on X such that $0 \to \mathrm{P}^\bullet_X \otimes \mathcal{O}_M \to 0$ is exact where $M \subset X_{\mathrm{red}}$ and $\dim M = r$. Then $0 = \mathrm{c}_i^{X_{\mathrm{red}}}(\mathcal{F}) \in \mathrm{H}^i(X, \Omega^i_{X_{\mathrm{red}}})$ for $1 \leq i \leq \min\{r, n-2\}$.*

Proof. Since $0 \to \mathrm{P}^\bullet_X \otimes \mathcal{O}_M \to 0$ is exact, $\mathrm{c}^M(\mathcal{F}) = 1$ by Property 5. From Lemma 1 above, it follows that $\forall\ 1 \leq i \leq \min\{r, n-2\}$, there exist classes $t_i \in \mathrm{H}^i(\Omega^i_{\mathbb{P}^n})$ such that

$$\beta(\mathrm{c}_i(\mathcal{F})) = \alpha(t_i).$$

Let $\theta : \mathrm{H}^i(\Omega^i_{X_{\mathrm{red}}}) \to \mathrm{H}^i(\Omega^i_M)$ be the natural map. Then $\theta\beta(\mathrm{c}_i(\mathcal{F})) = \mathrm{c}_i^M(\mathcal{F}) = 0$ for $i > 0$. Thus, $\theta\alpha(t_i) = 0$. But, $\theta\alpha = \gamma$ and by Lemma 2, we get that $t_i = 0$ for $1 \leq i \leq \min\{r, n-2\}$ and thus

$$\mathrm{c}_i^{X_{\mathrm{red}}}(\mathcal{F}) = \beta\, \mathrm{c}_i(\mathcal{F}) = 0$$

for $1 \leq i \leq \min\{r, n-2\}$. □

3. Generalised Quadrics

In this section, we apply the results of the previous section to show that generalised quadrics in $\mathbb{P}^{2n+1}$ for $n \geq 1$ are reduced.

Let $Q \subset \mathbb{P}^{2n+1}$ denote the generalised quadric given by the equation $\sum_{i=0}^n f_i g_i = 0$. Let

$$Z := Q \cap (f_1 = \cdots = f_n = 0)$$

$$L_1 := Q \cap (f_0 = \cdots = f_n = 0)$$

$$L_2 := Q \cap (g_0 = f_1 = \cdots = f_n = 0)$$

Then $Z = L_1 \cup L_2$ and we have an exact sequence

$$0 \to \mathcal{O}_{L_2}(-\deg f_0) \to \mathcal{O}_Z \to \mathcal{O}_{L_1} \to 0$$

Furthermore, Z is a complete intersection of n ample divisors on Q, L_i for $i = 1, 2$ are local complete intersection subschemes in Q of codimension (and dimension) n.

Theorem 1. *The generalised quadric Q is reduced.*

Proof. If Q is not reduced, let X be an irreducible component of Q which is not reduced and let X_{red} denote the subscheme X with the reduced structure. Thus $\sum f_i g_i = f^r f'$ with f an irreducible polynomial, $r > 1$ where $f^r = 0$ defines X and $f = 0$ defines X_{red}.

Let $Z' = Z \cap X$, $L'_i = L_i \cap X$. It is easy to see that Z' is a complete intersection in X by $f_i, i > 0$. We consider the Koszul resolution of $\mathcal{O}_{Z'}$ on X:

$$0 \to \mathcal{O}_X(-\sum_i a_i) \to \cdots \to \oplus_i \mathcal{O}_X(-a_i) \to \mathcal{O}_X \to \mathcal{O}_{Z'} \to 0$$

By a formal computation, it follows that

$$\mathrm{c}_n(\mathcal{O}_{Z'}) = ah^n \in \mathrm{H}^n(\Omega^n_X)$$

where $a = (-1)^{n-1}(n-1)!\ (\Pi_i a_i) \neq 0$.

On the other hand, since L'_i are local complete intersections in X, there exist finite resolutions by vector bundles over X for the sheaves $\mathcal{O}_{L'_i}$:

$$0 \to \mathrm{P}^\bullet_i \to \mathcal{O}_{L'_i} \to 0.$$

We have an exact sequence,

$$0 \to \mathcal{O}_{L'_2}(-d) \to \mathcal{O}_{Z'} \to \mathcal{O}_{L'_1} \to 0$$

where $d = \deg f_0$ which gives by Property 2 that

$$\mathrm{c}(\mathcal{O}_{Z'}) = \mathrm{c}(\mathcal{O}_{L'_1})\,\mathrm{c}(\mathcal{O}_{L'_2}(-d)) \in \mathrm{H}^\bullet(\Omega^\bullet_X).$$

Let M_1 be the subscheme defined by the vanishing of $g_0, \ldots, g_n$ in X_{red}. Then $\dim M_1 = n$ and since $L_1 \cap M_1 = \emptyset$, we get, $0 \to \mathrm{P}^\bullet_1 \otimes \mathcal{O}_{M_1} \to 0$ is exact. Since $\dim M_1 = n = \min\{n, 2n+1-2\}$, by Lemma 3, we see that $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{L'_1}) = 1 + x$, where $x \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\text{red}}})$. A similar argument with L_2 and M_2 (which is defined by the vanishing of $f_0, g_1, \ldots, g_n$ on X_{red}) gives, $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{L'_2}(-d)) = 1 + y$. Thus by Property 6, $\mathrm{c}^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 1 + z$ with $z \in \oplus_{i>n} \mathrm{H}^i(\Omega^i_{X_{\text{red}}})$. In particular, we see that $c_n^{X_{\text{red}}}(\mathcal{O}_{Z'}) = 0$. But, we have seen that this is the image of ah^n for $a \neq 0$, h the class of hyperplane section. By Lemma 2, this is a contradiction. $\square$

4. Appendix

The purpose of this appendix is to prove the following theorem which is folklore, but we thought we will give a proof for completeness.

Theorem 2. *Let X be a projective scheme of dimension $r \geq 1$, $h \in \mathrm{H}^1(X, \Omega^1_X)$ the class of a hyperplane section. Then h^r in $\mathrm{H}^r(X, \Omega^r_X)$ is not zero.*

Since we plan to show some class is not zero, we will usually not worry about correctly identifying cohomology classes and allow ourselves the liberty of multiplying by non-zero constants. In other words, cohomology groups will not be canonically identified and we will allow ourselves choice of bases. We first prove a slightly stronger theorem for $\mathbb{P}^r$.

Let $h \in \mathrm{H}^1(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r})$ be the class of a hyperplane. We will assume the well known facts that $h^i \in \mathrm{H}^i(\mathbb{P}^r, \Omega^i_{\mathbb{P}^r})$ generates this one dimensional vector space (in particular $h^i \neq 0$) and $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^r}(1))$ is a non-zero multiple of h.

Theorem 3. *Let H be a hyperplane section of $\mathbb{P}^r$ with $r \geq 2$. Then we have a canonical isomorphism $\alpha : \mathrm{H}^{r-1}(H, \Omega^{r-1}_H) \to \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r})$ and $\alpha(h^{r-1}) = ch^r$, c a non-zero constant where each of the h's is the class of an appropriate hyperplane. (If we had made the correct identifications, then $c = 1$).*

Proof. We have a canonical exact sequence,

$$0 \to \Omega^r_{\mathbb{P}^r} \to \Omega^r_{\mathbb{P}^r}(H) \to \Omega^{r-1}_H \to 0.$$

This gives, by taking cohomologies an isomorphism

$$\alpha : \mathrm{H}^{r-1}(H, \Omega^{r-1}_H) \to \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}),$$

using the fact that $\mathrm{H}^i(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}(H)) = 0$ for $i = r-1, r$. □

Lemma 4. *Let C be a non-singular projective curve and let L be an ample line bundle. Then $l = \mathrm{c}_1(L) \in \mathrm{H}^1(C, \Omega^1_C)$ is not zero.*

Proof. It is clear that we may replace l by nl for any $n > 0$ and thus we may assume that L is very ample. This gives, by taking two general sections of L, a morphism $f : C \to \mathbb{P}^1$ with $f^*(\mathcal{O}_{\mathbb{P}^1}(1)) = L$. Since

$$l = \mathrm{c}_1(L) = \mathrm{c}_1(f^*(\mathcal{O}_{\mathbb{P}^1}(1))) = f^*(\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1}(1))),$$

it suffices to prove that $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}^1}(1)) \neq 0$ which we have assumed and that $f^* : \mathrm{H}^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) \to \mathrm{H}^1(C, \Omega^1_C)$ is injective. The second statement is obvious, since the natural map $\Omega^1_{\mathbb{P}^1} \to f_*\Omega^1_C$ splits. □

Proof of Theorem 2. Let $Y \subset X$ be a reduced irreducible closed subvariety of dimension r. We have a natural map $\mathrm{H}^r(X, \Omega^r_X) \to \mathrm{H}^r(Y, \Omega^r_Y)$. Thus it suffices to prove the theorem for Y, since h^r goes to h^r. Thus we may assume that X is integral. Similarly, we may replace X by its normalization and thus assume that X is normal. Proof is by induction on r where the case $r = 1$ is treated in lemma 4.

For the induction step we proceed as follows. If h is the class of the ample line bundle H, we may clearly replace H by $nH, n > 0$. Thus we may assume the following. $\mathrm{H}^i(X, \Omega^r_X(H)) = 0$ for $i = r-1, r$, since $r \geq 2$. Further, we have a section $Y \in |H|$ which is integral and normal and the multiplication map $Y : \Omega^r_X \to \Omega^r_X(H)$ is injective. Let $\mathcal{E}$ be the cokernel of this map. We may also assume that we have a finite map $f : X \to \mathbb{P}^r$ such that $f^*(\mathcal{O}_{\mathbb{P}^r}(1)) = H$ and a section $s \in \mathrm{H}^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1))$ such that f^*s corresponds to Y. Let us denote the hyperplane $s = 0$ by L. By our assumption, we see that the map $\mathrm{H}^{r-1}(Y, \mathcal{E}) \to \mathrm{H}^r(X, \Omega^r_X)$ is an isomorphism. We also see that on the smooth points of Y, $\mathcal{E} \cong \Omega^{r-1}_Y$. This says that the double dual of $\mathcal{E}$ and Ω^{r-1}_Y are isomorphic. We denote the double dual by $\mathcal{F}$. Thus we have maps $\mathcal{E} \to \mathcal{F}$ and $\Omega^{r-1}_Y \to \mathcal{F}$ which are isomorphisms on the open subset of smooth points. Since the codimension of the singular locus is at least 2, we see that

$$\mathrm{H}^{r-1}(Y, \mathcal{E}) \cong \mathrm{H}^{r-1}(Y, \mathcal{F}) \cong \mathrm{H}^{r-1}(Y, \Omega^{r-1}_Y).$$

Using f, we have a commutative diagram,

$$\begin{array}{ccc} \mathrm{H}^{r-1}(L, \Omega^{r-1}_L) & \xrightarrow{\cong} & \mathrm{H}^r(\mathbb{P}^r, \Omega^r_{\mathbb{P}^r}) \\ \downarrow f^* & & \downarrow f^* \\ \mathrm{H}^{r-1}(Y, \mathcal{E}) & \xrightarrow{\cong} & \mathrm{H}^r(X, \Omega^r_X) \end{array}$$

We have the natural map $\mathrm{H}^{r-1}(L, \Omega^{r-1}_L) \xrightarrow{f^*} \mathrm{H}^{r-1}(Y, \Omega^{r-1}_Y)$ and the class of h^{r-1} goes to a non-zero element by induction. But the latter group is isomorphic to $\mathrm{H}^{r-1}(Y, \mathcal{E})$ and thus h^{r-1} goes to a non-zero element in this group and then by the above isomorphism, its image in $\mathrm{H}^r(X, \Omega^r_X)$ is non-zero. Now, following h^{r-1} via the other branch of the commutative diagram, we see that $h^r \neq 0$ in $\mathrm{H}^r(X, \Omega^r_X)$ by theorem 3.

□

References

[1] M. F. Atiyah, *Complex Analytic Connections in Fibre Bundles*, Trans. Amer. Math. Soc. 85 (1957), 181–207.

[2] G. Ellingsrud, L. Gruson, C. Peskine and S. A. Strømme, *On the normal bundle of curves on smooth projective surfaces*, Inventiones Mathematicae, 80 (1985) 181–184.

[3] Illusie, Luc, *Complexe cotangent et déformations. I, II*, Lecture Notes in Mathematics, Vols. 239 and 283. Springer-Verlag.

[4] Mohan Kumar, N., Peterson, Chris; Prabhakar Rao, A., *Construction of low rank vector bundles on $\mathbb{P}^4$ and $\mathbb{P}^5$*, J. Algebraic Geom. 11 (2002), no. 2, 203–217.

[5] N. Mohan Kumar, A. P. Rao, G.V.Ravindra, *Notes on Hodge style Chern classes for vector bundles on schemes*, available at `http://www.math.wustl.edu/~kumar`

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