A NOTE ON CANCELLATION OF REFLEXIVE MODULES

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1. Introduction

By the Quillen-Suslin theorem [Qui76, Sus76], we know that projective modules over a polynomial ring over a field are free. One way of saying this is, that if two projective modules of the same rank are stably isomorphic, then they are isomorphic. That, projective modules of given rank over polynomial rings are stably isomorphic was well known at the time Quillen and Suslin proved their theorems and this result is usually attributed to Grothendieck. This result can also be deduced from Hilbert Syzygy Theorem. This note tries to answer whether a similar cancellation occurs for reflexive modules. The question was specifically raised by M. P. Murthy. I thank him for raising this question and for the innumerable discussions which ensued. In this note we show that, in general, reflexive modules are not cancellative, without further assumptions. These assumptions under which cancellation does take place are explained in the theorem in the next section, Theorem 1.

2. A case where cancellation is true

As usual, we say that a module M over a ring A is cancellative, if for some finitely generated free module F over A and a module N, if $F \oplus M \cong F \oplus N$, then $M \cong N$. The main theorem we will prove is the following:

Theorem 1. *Let R be an affine domain over an algebraically closed field of characteristic zero, M a reflexive module over R of finite homological dimension such that M is locally free outside a finite set of closed points of R. Further assume that* $\operatorname{rank} M \geq \dim R$. *Then M is cancellative.*

We start with some lemmas. We will not state the most general versions, but just what we need.

Lemma 1. *Let A be an affine algebra over an infinite field k of dimension n and Q a projective module (of constant rank) over A. Let $a \in A$ and let $I \subset A$ be an ideal of height n . Then there exists an element $f = \lambda + a + x$, with $x \in I$ and $0 \neq \lambda \in k$ such that Q_f is free over A_f.*

Proof: Since I is of height n, Q is free when we semi-localise at the finitely many maximal ideals containing I. Thus we may find an $s \in A$, comaximal with I such that Q_s is A_s-free. Since the base field is infinite, for a general $0 \neq \lambda \in k$, $a + \lambda$ is comaximal with I. Since s is a unit modulo I, we can find a $t \in A$ such that $st \equiv a + \lambda \pmod{I}$. Notice that Q_{st} is clearly free over A_{st}. So, letting $f = st$, we have, $f = \lambda + a + x$ with $x \in I$.

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Corollary 1. *Notation being the same as in the lemma, there exists an $x \in I$ such that either $a+x$ is a unit in A or $Q/(a+x)Q$ is a free module over $A/(a+x)A$.*

Proof: From the lemma, we have $f-\lambda = a+x$ with $0 \neq \lambda \in k$ and Q_f is A_f free. But, then $Q/(f-\lambda)Q$ is clearly free over $A/(f-\lambda)A$, unless $f-\lambda$ is a unit.

In the following two lemmas, A will be an integral domain, M a finitely generated torsion-free A-module, $0 \neq a \in A, m \in M$ such that M/aM is a projective A-module of rank equal to the rank of $M = n$ over A/aA and the image of m in M/aM is a unimodular element of M/aM. The conditions imply (by an easy local checking), that the map $A \xrightarrow{(a,m)} A \oplus M$ is a split inclusion.

Lemma 2. *Let $m' \in M$ and $d \in \mathbb{N}$. Then,*

$$A \oplus M/(a^d, m)A \cong A \oplus M/(a^d, m+am')A.$$

The proof is essentially the same as in [Kum97, lemma 2] and is just a slight modification of a result of Suslin [Sus77b].

Proof: Consider $B = A[t]$, polynomial ring in one variable over A and let us consider the module, $N = B \oplus M[t]/(a^d, m+atm')$. We will show that N is an extended module and then putting $t = 0, 1$, we would be done.

The hypotheses on a, m imply that $B \oplus N \cong B \oplus M$. So, if $n = 1$, we are done by taking determinants. So, we will further assume that $n \geq 2$.

To check that N is extended, by Quillen's theorem [Qui76], we need to show this only locally. At maximal ideals not containing a, clearly $N \cong M[t]$. At maximal ideals containg a, by hypothesis M is free with m as part of a basis. So choose a basis, $m = m_1, m_2, \ldots, m_n$. Write $m' = \sum c_i m_i$. Then with respect to this basis, the vector $(a^d, m+atm')$ is, $(a^d, 1+ac_1t, ac_2t, \ldots, ac_nt)$. Since $a^d, 1+ac_1t$ generate the unit ideal, we can change ac_2t (which exists, since $n \geq 2$) by elementary transformation to 1 and thus N is free at such a maximal ideal.

Next we prove a crucial lemma, which is essentially a slight generalisation of a theorem of Suslin [Sus77a].

Lemma 3. *Notation being as above, assume further that M/aM is a free module and the image of m in M/aM is part of a free basis of M/aM. Then,*

$$A \oplus M/(a^n, m) \cong M.$$

Proof: Since M/aM is free over A/aA with image of m as a part of a free basis, we may choose $m = m_1, m_2, \ldots, m_n \in M$ such that their images in M/aM form a free basis. Consider $B = A[t]$ and maps, $\phi(t) : B^n \to B^n$ and $\psi : B^n \to M$ given as follows. The map ψ is just sending a basis $\{e_i\}$ of B^n to the m_i's. The map $\phi(t)$ is given by the $n \times n$ matrix, which has a for its diagonal entries and t on the subdiagonal, with zero elsewhere. That is, $\phi(t)(e_1) = ae_1$ and $\phi(t)(e_i) = te_{i-1} + ae_i$ for $i > 1$. Consider $N = B^n \oplus M[t]/K$ where K is the image of B^n under the map $(\phi(t), \psi)$.

First, I claim that N is extended. Again, by Quillen's theorem *loc. cit*, suffices to do this locally on A. At a maximal ideal not containg a, $\phi(t)$ is an isomorphism and thus $N \cong M[t]$. For a maximal ideal containing a, by choice, ψ is an isomorphism and thus $N \cong B^n$, in particular extended. Thus N is extended. Also, notice that a is a non-zero divisor on N, since it is free at maximal ideals containing a and A is an integral domain. Thus, we have $N_{|t=0} = N_0 \cong N_1 = N_{|t=1}$.

Next let us look at N_0. Then, we have $M \subset N_0$ and $N_0/M \cong (A/aA)^n$. Since N_0 is a projective module of rank n at primes containing a, we see that $M = aN_0$ and since a is a non-zero divisor on N_0, $M \cong N_0$.

Finally, let us look at N_1. If we let $e_1' = e_1 - ae_2 + a^2e_3 - \cdots$, then $\phi(1)(e_1') = (-1)^{n-1}a^ne_n$. Consider the projection $\pi : A^n \to A^{n-1}$, to the first $n-1$ factors. Then $\pi \circ \phi(1)$ is onto and the kernel is generated by e_1'. Thus we can identify N_1 as the cokernel of the map

$$Ae_1' \stackrel{(\phi(1),\psi)}{\longrightarrow} Ae_n \oplus M,$$

and $(\phi(1),\psi)(e_1') = ((-1)^{n-1}a^ne_n, m_1 - am_2 + a^2m_3 - \cdots)$. Now, by the previous lemma, we see that $N_1 \cong A \oplus M/(a^n, m)$ and since $N_1 \cong N_0 \cong M$, we are done.

Proof of the Theorem: With the notation as in the theorem, we have an inclusion $R \stackrel{(a,m)}{\longrightarrow} R \oplus M$, which is split and the cokernel is N. We wish to show that $M \cong N$. We will use the following transvections, which do not change the situation. For any $\phi : M \to R$, we may replace (a,m) by $(a + \phi(m), m)$. Similarly, for any $m' \in M$, we may replace (a,m) by $(a, m + am')$.

That, (a,m) gives a split inclusion implies there exists a homomorphism $\phi : M \to R$ and $b \in R$ such that $ab + \phi(m) = 1$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the maximal ideals outside which M is locally free. Let

$$X = \{\mathfrak{p} \in \operatorname{Spec} R \mid a \notin \mathfrak{p} \text{ and } \mathfrak{p} \neq \mathfrak{m}_i\}.$$

On this open set X, M is locally free and aM generates M. So for a general choice of $m' \in M$, $m + am'$ vanishes at only a subset $Z \subset X$ of codimension $\geq$ rank M by Bertini's theorem. By assumption on the rank of M, this codimension is at least the dimension of R. Thus the map $m + am' : M^* \to R$ has image height at least $\dim R$ restricted to X. Also, since m is unimodular modulo aM, $m + am'$ is unimodular at primes containig a and thus we see that the image of $m + am' : M^* \to R$ has height at least $\dim R$ and comaximal with a. We rename $m + am'$ as m since this is an allowed transvection for our result and thus we may assume that $m(M^*) = I \subset R$ has height at least $\dim R$ and I is comaximal with a. Let $J = I \cap_i \mathfrak{m}_i$. We will arrange a so that $a \notin \mathfrak{m}_i$ for all i. Notice that for any $\psi \in M^*$, $a + \psi(m)$ is comaximal with I. Let $\phi \in M^*$ be such that a is comaximal with $\phi(m)$ as before. Though now we have a new m, the same ϕ actually works for this m too, though it is not important.

We may assume that $a \in \mathfrak{m}_i$ for $1 \leq i \leq p \leq r$ and $a \notin \mathfrak{m}_i$ for $p < i \leq r$, possibly after rearranging the $\mathfrak{m}_i$'s. Choose $x \in \cap_{i>p}\mathfrak{m}_i - \cup_{i=1}^p \mathfrak{m}_i$. Then we may replace (a,m) by $(a + x\phi(m), m)$. Then we see that $a + x\phi(m)$ is not in any one of the above maximal ideals. Thus we may assume that a is comaximal with J.

Since characteristic of the field is zero and it is algebraically closed, by Chinese remainder theorem, we can find a $b \in R$ such that $a \equiv b^d \pmod J$, where d is the rank of M. Since $a = b^d + y$ with $y \in J$ and since $J \subset I$, there exists a $\psi \in M^*$ such that $\psi(m) = y$. So, we may replace a with b^d. Notice that since $y \in J$ and a is comaximal with J, the same holds for b.

Let K_0 be the Grothendieck group of projective modules, which is the same as the Grothendieck group of modules of finite projective dimension. Thus $[M] \in K_0(R)$ by hypothesis and so we can write $[M] = [P] - [F]$ for some projective module P over R and F, a free module over R. Then we have an $x \in J$ so that $P/(b+x)P$ is free by corollary 1. Let $c = b + x$. Then, $c^d = b^d + xz$ for some $z \in R$ and since $x \in J$, there exists a $\psi \in M^*$ with $\psi(m) = xz$. So we can replace (b^d, m) with (c^d, m). Since M is projective at all maximal ideals containing c, we see that M/cM is projective and since c is a non-zero divisor in M, we see that $[M/cM] = [P/cP] - [F/cF]$ in $K_0(R/cR)$. By choice of c, P/cP is free over R/cR and thus, the projective module M/cM is stably free over R/cR. It has rank $\geq \dim R > \dim R/cR$. So, M/cM is free over R/cR by Bass' cancellation theorem [Bas68]. Also, the image of m in M/cM is unimodular and so by Suslin's theorem [Sus77b], we see that the image of m is part of a free basis of M/cM. Now, lemma 3 finishes the proof.

3. Reflexive modules over polynomial rings

Since reflexive modules over polynomial rings in one or two variables are free (the two variable case was originally proved by C. S. Seshadri, [Ses58]), we will assume that we are over a polynomial ring in at least three variables. Also, an example in n variables which is not cancellative give an example over polynomial rings in $k \geq n$ variables, by extending the ring and tensoring the module.

3.1. **Four variable case.** Let $R = k[x, y, z, t]$ be a polynomial ring in 4 variables. Let $v_1 = (x, y, zt - 1)$ and $v_2 = (x, yz, zt - 1)$ be two vectors giving rise to presentations of two modules M_1, M_2. Then M_i's are reflexive rank two modules over R. I claim that they are non-isomorphic but $M_1 \oplus R$ is isomorphic to $M_2 \oplus R$.

Notice that $v_1 R = v_2 R = I$, a complete intersection height three ideal in R. We have exact sequences,

$$0 \to M_i^* \to R^3 \to I \to 0.$$

We have a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & M_1^* & \to & R^3 & \to & I & \to & 0 \\ & & \uparrow f & & \uparrow \psi & & \uparrow \mathrm{Id} & & \\ 0 & \to & M_2^* & \to & R^3 & \to & I & \to & 0 \end{array}$$

where ψ is the diagonal matrix, $[1, z, 1]$. Thus we get an exact sequence,

$$0 \to M_2^* \to M_1^* \to R/zR \to 0.$$

Now by Schanuel's lemma type argument, we get an exact sequence,

$$0 \to R \to R \oplus M_2^* \to M_1^* \to 0.$$

Dualising this and noting that $\mathrm{Ext}^1(M_i^*, R) = R/I$, we see that,

$$0 \to M_1 \to M_2 \oplus R \to R \to 0$$

is exact and since R is free, it is split exact. Thus, $M_1 \oplus R \cong M_2 \oplus R$.

Now assume that $M_1 \cong M_2$. Then we get a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & R & \to & R^3 & \to & M_1 & \to & 0 \\ & & \downarrow a & & \downarrow \phi & & \downarrow \cong & & \\ 0 & \to & R & \to & R^3 & \to & M_2 & \to & 0 \end{array}$$

With respect to the given bases, we can identify ϕ as a 3×3 matrix. So, we see that $\det \phi = ca$ where $c \in k$ is a non-zero constant. Dualising, we see that a is a unit modulo I. Now we have the dual picture as follows:

$$\begin{array}{ccccccccc} 0 & \to & M_1^* & \to & R^3 & \to & I & \to & 0 \\ & & \uparrow \cong & & \uparrow \phi^* & & \uparrow a & & \\ 0 & \to & M_2^* & \to & R^3 & \to & I & \to & 0 \end{array}$$

Let us go modulo I. Then I/I^2 is a free module of rank 3 over R/I and we have,

$$\begin{array}{ccc} (R/I)^3 & \to & I/I^2 \\ \uparrow \phi^* & & \uparrow a \\ (R/I)^3 & \to & I/I^2 \end{array}$$

We notice that all the maps are now isomorphisms. So, let us compute ϕ^* modulo I with respect to the given bases. One immediately sees that ϕ^* is the diagonal matrix, $[a, az, a]$. Thus the determinat of ϕ^* is a^3z modulo I. But this is equal to ca since $\det\phi = \det\phi^*$. So, we get $a^3z \equiv ca \bmod I$ or $z \equiv ca^{-2} \bmod I$. But since $R/I = k[z, z^{-1}]$, such an equation cannot hold. Thus we see that M_1 and M_2 are not isomorphic.

One of the natural questions that can be raised is whether cancellation does hold if the ranks of the modules are sufficiently large. But, a modification of the above example shows that it is not the case.

For this, take

$$v_1 = (x^n, x^{n-1}y, \ldots, y^n, zt-1) \,\text{and}\, v_2 = (zx^n, x^{n-1}y, \ldots, y^n, zt-1)$$

and consider the corresponding reflexive modules M_1, M_2. These have rank $n+1$ and as before, it is easy to show that they are isomorphic, if we add a free module of rank one to both sides, since the ideal $Rv_1 = Rv_2 = I$ is of codimension 3 and R/I is Cohen-Macaulay. If they were isomorphic, exactly as before, restricting to R/J, where $J = (x, y, zt-1) = \operatorname{rad} I$ we get an equation, $ca = a^{n+2}z$ in $R/J = k[z, z^{-1}]$, where c is a non-zero constant in k and $a \in R$, is a unit in R/J. This leads to a contradiction.

The above examples show that we cannot drop the hypothesis of *locally free outside a finite set of points* in our theorem.

3.2. **Three variable case.** In three variable polynomial rings, any reflexive module is free outside a finite set of points. So, if the base fileld is algebraically closed and the rank is at least three, our theorem will imply that they are cancellative. Here we construct examples when the base field is not algebraically closed or if it is of positive characteristic, of examples which are not cancellative.

3.2.1. *Positive characteristic case.* In this section, we assume that our ring $R = k[x, y, z]$ is a polynomial ring in three variables and the charactersitic of $k = p > 0$. Then, we will construct examples of reflexive modules of rank p which are stably isomorphic, but not isomorphic. The method is essentially the same as above.

Consider, $v_1 = (x^{p(p-1)}, x^{p(p-2)}y, \ldots, y^{p-1}, z)$ and v_2 the same as v_1, except we replace y^{p-1} with $(1+x)y^{p-1}$. Then notice that $Rv_1 = Rv_2 = I$ a height three ideal, R/I is Cohen-Macaulay, and if we denote by $J = (x^p, y, z)$, then I/JI is a free module over R/J, of rank $p+1$. Now the argument is the same as above and if the corresponding modules are isomorphic, we get that the element $1+x$ has a p^{th} root in $R/J = k[x]/x^p$, which is impossible.

3.2.2. *Characteristic zero case.* Now let us assume that $R = k[x, y, z]$ where k is a field of characteristic zero and let $n \geq 2$ be an integer. Further assume that there exists a finite extension L of k such that $(L^*)^n k^* \neq L^*$ and pick an element $c \in L^* - (L^*)^n k^*$. Using c, as before we will construct rank n reflexive modules over R which are stably isomorphic, but not isomorphic. Let $p(x)$ be the irreducible polynomial such that $L = k[x]/p(x)$ and let $q(x)$ be so chosen so that its image in L via the above map is c. Then, we consider the two vectors,

$$v_1 = (p^{n-1}, p^{n-2}y, \ldots, y^{n-1}, z), v_2 = (p^{n-1}, p^{n-2}y, \ldots, y^{n-1}, qz).$$

One checks as before that the first syzygyies of these two vectors are rank n reflexive modules, which are stably isomorphic, but not isomorphic.

To obtain such examples of fields, we can take for example, $k = \mathbb{Q}$ and $L = \mathbb{Q}(\theta)$ where $\theta = \sqrt[n]{2}$. If $(L^*)^n k^* = L^*$, then we get $\theta = a^n b$ where $a \in L^*$ and $b \in k^*$. Taking norms we see that either 2 or -2 is an n^{th} power in k, which is impossible unless $n = 1$.

So, we see that the hypothesis of algebraically closed field, charactersitic zero and locally free outside a finite set of closed points are all essential for our theorem. Since we have examples [Kum85] of stably free non-free modules of rank $= \dim R - 2$, the condition on the rank is more or less necessary, except possibly when it is one less than the dimension of the ring. The condition on finite homological dimension on the other hand, does not seem essential, but we have neither a proof nor a counter example.

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