

Math 131 , Spring 2004
Final Exam Solutions

Name: _____ ID# _____

No graphing calculators or calculators with a CAS are allowed. You may use a scientific calculator. Be sure your calculator is set for “radians”, not “degrees”, if you do any calculus computations with trig functions. For Part I, please mark your answer on the answer card. For Part II, please solve the problems in the space provided.

Part I, Multiple Choice, 4 points/problem:

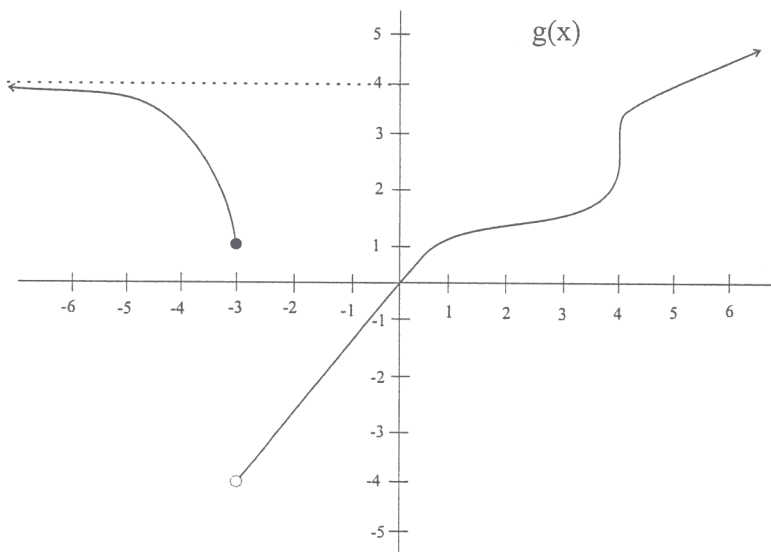
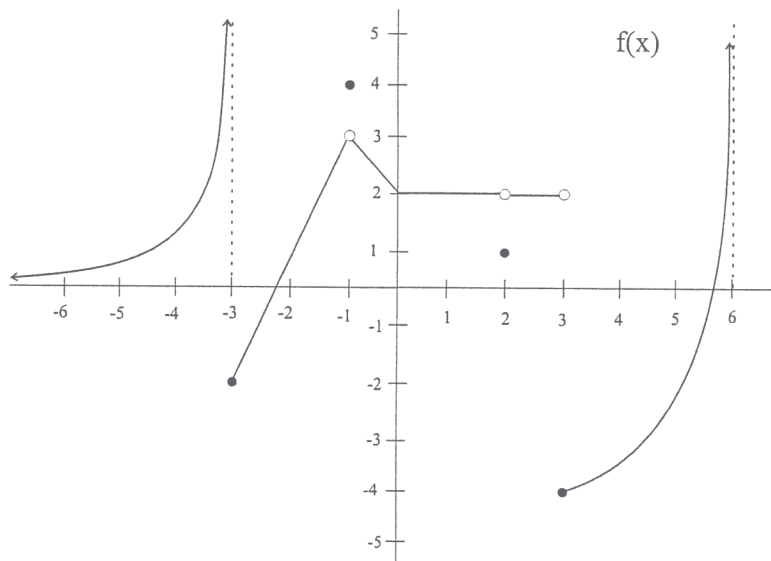
1. Which of the following is $f'(x)$ if $f(x) = \frac{\arctan x}{1+x^2}$. (Note that $\arctan x$ is the same as $\tan^{-1}(x)$.)

- A) $\frac{1}{2x(1+x^2)}$ B) $\frac{1-2x \arctan x}{(1+x^2)^2}$ C) $\frac{2x \arctan x - 1}{(1+x^2)^2}$ D) $\frac{1}{2x(1+x^2)^2}$
- E) $\frac{1}{(1+x^2)^2}$ F) $\frac{(1+x^2) \sec^2 x - 2x \arctan x}{(1+x^2)^2}$ G) $-\frac{2x}{(1+x^2)^2}$ H) $\frac{\sec^2 x}{2x}$
- I) $x \arctan(1+x^2)$ J) $\frac{1}{(1+x^2)} - 2x$

Solution: Using the quotient rule,

$$f'(x) = \frac{(1+x^2) \left(\frac{1}{1+x^2} \right) - \arctan x(2x)}{(1+x^2)^2} = \frac{1-2x \arctan x}{(1+x^2)^2}.$$

For questions 2 - 11, please refer to the graphs of $f(x)$ and $g(x)$ given below. Questions 2 - 6 (worth 2 points each) are “fill in the blank”, where you will select your answers from those listed below the graphs. Questions 7 - 11 (worth 2 point each) are true/false questions.



The “fill in the blank” answer choices are:

- A) -4 B) -3 C) -2 D) -1 E) 0
 F) 1 G) 2 H) 3 I) 4 J) ∞

2. $\lim_{x \rightarrow 6^-} \frac{g(x)}{f(x)} = 0$. (i.e. answer E) **Solution:** $g(x) \rightarrow 4.5$ and $f(x) \rightarrow \infty$ as $x \rightarrow 6^-$, so $\frac{g(x)}{f(x)} \rightarrow 0$

3. $\lim_{x \rightarrow 0^+} \frac{f(x)}{g(x)} = \infty$. (i.e. answer J) **Solution:** $f(x) \rightarrow 2$ and $g(x) \rightarrow 0$ as $x \rightarrow 0^+$, so $\frac{f(x)}{g(x)} \rightarrow \infty$.

4. $f''(-1.5) = 0$. (i.e. answer E) **Solution:** $f(x)$ is a line segment on $(-3, -1)$, so it has no concavity.

5. $\lim_{x \rightarrow -3^+} g(x) = -4$. (i.e. answer A)

6. $\lim_{x \rightarrow -1} f(x) = 3$. (i.e. answer H)

The following 5 questions are true/false and still refer to the functions $f(x)$ and $g(x)$, whose graphs are shown on the previous page.

7. f is differentiable at $x = -1$.

A) True

B) False

Solution: f is not continuous at $x = -1$, so it can't be differentiable there.

8. f is continuous at $x = -1$.

A) True

B) False

Solution: $f(-1)$ is not defined, so the definition of continuity is not met.

9. $x = 4$ is a critical value for $g(x)$.

A) True

B) False

Solution: $g'(-4)$ does not exist since g has a vertical tangent line at $x = -4$.

10. $g''(x) < 0$ on $(-\infty, -3)$.

A) True

B) False

Solution: g is concave down on $(-\infty, -3)$.

11. $f'(x) > 0$ on $(-\infty, -3)$.

A) True

B) False

Solution: f is increasing on $(-\infty, -3)$.

12. Find the absolute minimum value of the function $f(x) = x^3 - 12x + 1$ on the interval $[-3, 5]$.

A) 0 B) -1 C) -2 D) -3 E) -9

F) -15 G) -18 H) -22 I) -25 J) -50

Solution: Candidates for absolute extrema are the critical values and the endpoints of the interval (which are $x = -3, 5$ in this case). Now,

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2),$$

so the critical values of $f(x)$ are $x = \pm 2$. Then $f(-3) = 10$, $f(-2) = 17$, $f(2) = -15$, and $f(5) = 66$, so the absolute minimum is -15 .

13. Find $\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{2 \sin(3x)}$.

A) ∞ B) 4 C) 2 D) $\frac{4}{3}$ E) 1

F) $\frac{1}{2}$ G) $\frac{2}{3}$ H) -1 I) 0 J) $-\pi$

Solution: The given limit is an indeterminate form of type " $\frac{0}{0}$ ", so we use L'Hôpital's Rule.

$$\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{2 \sin(3x)} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 0} \frac{4e^{4x}}{6 \cos(3x)} = \frac{4e^0}{6 \cos(0)} = \frac{4}{6} = \frac{2}{3}.$$

14. Suppose a particle moving in a straight line has velocity function $v(t) = t - 2$ (in meters per second) for $0 \leq t \leq 6$. What is the particle's displacement?

- A) 0 meters B) 4 meters C) 6 meters D) 8 meters E) 10 meters
F) 12 meters G) 16 meters H) 18 meters I) 24 meters J) 36 meters

Solution: Let $s(t)$ be the position of the particle at time t . Then

$$\begin{aligned} \text{displacement} &= s(6) - s(0) = \int_0^6 s'(t) dt = \int_0^6 v(t) dt \\ &= \int_0^6 (t - 2) dt \\ &= \left. \frac{t^2}{2} - 2t \right|_0^6 \\ &= 18 - 12 \\ &= 6. \end{aligned}$$

15. What is the total distance travelled for the particle described in problem 14?

- A) 0 meters B) 36 meters C) 24 meters D) 18 meters E) 16 meters
F) 14 meters G) 12 meters H) 10 meters I) 6 meters J) 2 meters

Solution: First note that the particle moves in the forwards and backwards direction during the time interval $0 \leq t \leq 6$ since $v(t) > 0$ for $2 < t \leq 6$ and $v(t) < 0$ for $0 \leq t < 2$. Let T = total distance travelled. Then

$$\begin{aligned} T &= \int_0^6 |v(t)| dt = \int_0^6 |t - 2| dt = \int_0^2 (2 - t) dt + \int_2^6 (t - 2) dt \\ &= \left. 2t - \frac{t^2}{2} \right|_0^2 + \left. \frac{t^2}{2} - 2t \right|_2^6 \\ &= 2 + 8 \\ &= 10. \end{aligned}$$

16. For what values of x is the function $f(x) = x^3 - x$ both **increasing and concave down**?

A) no values of x

B) $x < -\frac{1}{\sqrt{3}}$ and $x > \frac{1}{\sqrt{3}}$

C) $0 < x < \frac{1}{\sqrt{3}}$

D) $0 < x < \frac{1}{3}$

E) $x > 0$

F) $-\frac{1}{3} < x < \frac{1}{3}$

G) $-\frac{1}{\sqrt{3}} < x < 0$

H) $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

I) $x > \frac{1}{\sqrt{3}}$

J) $x < -\frac{1}{\sqrt{3}}$

Solution: The function $f(x)$ is increasing when $f'(x) > 0$, and f is concave down when $f''(x) < 0$. We first find where f is increasing.

$$f'(x) = 3x^2 - 1 > 0 \quad \text{when} \quad x^2 > \frac{1}{3} \quad \text{when} \quad |x| > \frac{1}{\sqrt{3}}.$$

Notice that $|x| > \frac{1}{\sqrt{3}}$ means that $x > \frac{1}{\sqrt{3}}$ and $x < -\frac{1}{\sqrt{3}}$.

We next find where f is concave down.

$$f''(x) = 6x < 0 \quad \text{when} \quad x < 0.$$

Then f is increasing and concave down when $x < -\frac{1}{\sqrt{3}}$

Part II: These are the “free response” problems worth a total of 56 points. Write your answers on the test pages. Show your work neatly and cross out irrelevant scratch-work, false starts, etc.

Please put your NAME on each of the following pages, since they may be separated during grading. Remember to turn in your notecard with these pages if you wish to take part in the “Impress Me With Your Notecard . . . Win Great Prizes” contest.

Name: _____ Discussion Section: _____

17. (24 points) The theme of this question is integration.

a) Use the **definition** of the definite integral to evaluate $\int_0^4 (x + 1) dx$.

Solution: Please note that I have written this calculation performing only one step per line, so this computation is not as long or complicated as it may appear. Also note that $f(x) = x + 1$ and $\Delta x = \frac{4-0}{n} = \frac{4}{n}$.

$$\begin{aligned}
 \int_0^4 (x + 1) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (x_i^* + 1) \frac{4}{n} \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n (x_i^* + 1) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \sum_{i=1}^n \left(\frac{4i}{n} + 1 \right) \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\sum_{i=1}^n \frac{4i}{n} + \sum_{i=1}^n 1 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\frac{4}{n} \sum_{i=1}^n i + \sum_{i=1}^n 1 \right] \\
 &= \lim_{n \rightarrow \infty} \frac{4}{n} \left[\frac{4}{n} \left(\frac{n(n+1)}{2} \right) + n \right] \\
 &= \lim_{n \rightarrow \infty} \left[4 + 8 \left(\frac{n+1}{n} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \left[4 + 8 \left(1 + \frac{1}{n} \right) \right] \rightarrow 0 \\
 &= 4 + 8 \\
 &= 12.
 \end{aligned}$$

b) Evaluate $\int \frac{x^2 + \sqrt{x}(2^x + \sec^2 x) + 1}{\sqrt{x}} dx$.

Solution:

$$\begin{aligned}\int \frac{x^2 + \sqrt{x}(2^x + \sec^2 x) + 1}{\sqrt{x}} dx &= \int \left[\frac{x^2}{\sqrt{x}} + \frac{\sqrt{x}(2^x + \sec^2 x)}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right] dx \\ &= \int \left(x^{3/2} + 2^x + \sec^2 x + x^{-1/2} \right) dx \\ &= \frac{2}{5}x^{5/2} + \frac{2^x}{\ln 2} + \tan x + 2\sqrt{x} + C.\end{aligned}$$

c) Evaluate $\int \frac{1 + 6x}{\sqrt{2 + x + 3x^2}} dx$.

Solution: We use u -substitution with $u = 2 + x + 3x^2$. Then $du = (1 + 6x)dx$, so

$$\begin{aligned}\int \frac{1 + 6x}{\sqrt{2 + x + 3x^2}} dx &= \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du \\ &= 2\sqrt{u} + C \\ &= 2\sqrt{2 + x + 3x^2} + C.\end{aligned}$$

d) Evaluate $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

Solution: We use u -substitution with $u = \sqrt{x}$. Then

$$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx \implies 2du = \frac{dx}{\sqrt{x}}.$$

We will also change the limits of integration from the x -world to the u -world. When $x = 1$, $u(1) = \sqrt{1} = 1$, and when $x = 4$, $u(4) = \sqrt{4} = 2$. Then

$$\begin{aligned}\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^2 2e^u du = 2e^u \Big|_1^2 \\ &= 2(e^2 - e) \\ &= 2e(e - 1).\end{aligned}$$

Name: _____

Discussion Section: _____

18. (21 points) The theme of the following problem is derivatives and equations of tangent lines.

a) (6 points) Differentiate $f(x) = e^{\sin(x^2)}$.

Solution: This is a composition of three functions. We use the chain rule to differentiate.

$$\begin{aligned} f'(x) &= e^{\sin(x^2)} [\sin(x^2)]' \\ &= e^{\sin(x^2)} (\cos(x^2)) (x^2)' \\ &= e^{\sin(x^2)} (\cos(x^2)) (2x) \\ &= 2x \cos(x^2) e^{\sin(x^2)}. \end{aligned}$$

b) (5 points) Find $f'(x)$ if $f(x) = \ln\left(\frac{2x^2 \sin x}{x^6}\right)$. [Hint: There is an efficient solution to this problem, and a solution that makes for not very pleasant differentiation.]

Solution: We could immediately start using the chain rule, but that will not make for a pleasant calculation (because we would also have to use the quotient rule and the product rule). Instead, let's simplify first using log properties. Thus,

$$f(x) = \ln\left(\frac{2x^2 \sin x}{x^6}\right) = \ln(2x^2) + \ln(\sin x) - \ln(x^6) = x^2 \ln 2 + \ln(\sin x) - 6 \ln x.$$

Then

$$f'(x) = 2x \ln 2 + \frac{1}{\sin x} (\cos x) - \frac{6}{x} = 2x \ln 2 + \cot x - \frac{6}{x}.$$

c) (5 points) Find the equation of the tangent line to $x^3y^3 + x = 2$ at the point $(1, 1)$.

Solution: We first use implicit differentiation to find $\frac{dy}{dx}$.

$$\begin{aligned}\frac{d}{dx}(x^3y^3 + x) &= \frac{d}{dx}(2) \\ x^3 \left(3y^2 \frac{dy}{dx} \right) + y^3(3x^2) + 1 &= 0 \\ 3x^3y^2 \frac{dy}{dx} &= -(3x^2y^3 + 1) \\ \frac{dy}{dx} &= -\frac{(3x^2y^3 + 1)}{3x^3y^2}.\end{aligned}$$

Then the slope of the tangent line at the point $(1, 1)$ is

$$\frac{dy}{dx} = -\frac{(3x^2y^3 + 1)}{3x^3y^2} \bigg|_{(1,1)} = -\frac{(3(1)(1) + 1)}{3(1)(1)} = -\frac{4}{3}.$$

Lastly, the equation of the tangent line is

$$y - 1 = -\frac{4}{3}(x - 1) \implies y = -\frac{4}{3}x + \frac{7}{3} \implies y = -\frac{1}{3}(4x - 7).$$

d) (5 points) Find the equation of the tangent line to $y = f(x) = \int_1^{x^2} \sqrt{1+w^5} dw$ at $x = 1$.

Solution: Note that the function $f(x) = \int_1^{x^2} \sqrt{1+w^5} dw$ is a composition of two functions:

$$g(u) = \int_1^u \sqrt{1+w^5} dw \quad \text{and} \quad u = x^2,$$

so $f(x) = (g \circ u)(x) = g(x^2)$. To find $f'(x)$, we must use the chain rule. In detail,

$$f'(x) = g'(x^2)u'(x) = g'(x^2)(2x),$$

where

$$g'(u) = \sqrt{1+u^5} \quad \text{by the Fundamental Theorem of Calculus.}$$

Then

$$g'(x^2) = \sqrt{1+(x^2)^5} = \sqrt{1+x^{10}},$$

and

$$f'(x) = g'(x^2)(2x) = 2x\sqrt{1+x^{10}}.$$

Then the slope of the tangent line at $x = 1$ is $f'(1) = 2(1)\sqrt{1+1^{10}} = 2\sqrt{2}$. When $x = 1$, $y = f(1) = \int_1^1 \sqrt{1+w^5} dw = 0$ since the limits of integration are the same. Thus, the equation of the tangent line at the point $(1, 0)$ is

$$y = 2\sqrt{2}(x - 1).$$

Name: _____

Discussion Section: _____

19. (11 points) A potpourri of problems for your enjoyment. Remember to show all of your work.

a) (6 points) A ladder 50 feet long is leaning against a wall. The bottom of the ladder is sliding away from the wall at a constant rate of 2 ft/sec. At what speed is the top of the ladder sliding down the wall at the instant when the bottom of the ladder is 30 feet from the wall?

Solution: Let x be the horizontal distance from the base of the ladder to the wall.

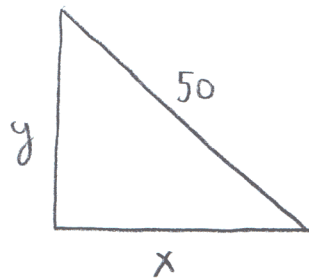
Let y be the vertical distance from the ground to the top of the ladder.

Given: $\frac{dx}{dt} = 2$ ft/sec

Find: $\frac{dy}{dt}$ when $x = 30$ feet.

Then

$$\begin{aligned}x^2 + y^2 &= 50^2 \\2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\2x \frac{dx}{dt} &= -2y \frac{dy}{dt} \\ \frac{dy}{dt} &= -\frac{x}{y} \frac{dx}{dt}.\end{aligned}$$



When $x = 30$, $y^2 = 50^2 - 30^2$, so $y = 40$. Then

$$\left. \frac{dy}{dt} \right|_{x=30} = -\frac{30}{40}(2) = -\frac{3}{2} \text{ ft/sec.}$$