MATH 233 MIDTERM EXAM 3

This exam consists of 12 multiple choice (machine-graded) problems, worth 5 points each (for a total of 60 points), and 2 pages of written (hand-graded) problems, worth a total of 40 points. No 3x5 cards or calculators are allowed.

$\boxed{\text{SOLUTIONS}}$

PART I: MULTIPLE CHOICE PROBLEMS

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. **Shade in the corresponding boxes below.** Also **print your name at the top of your card.**

(1) Find the minimum value of $f(x,y,z) = x^2 - 6x + y^2 - 8y + 7$ on the closed disk $\underbrace{x^2+y^2}_{g} \le 1$.

(A) -10
(B) -6
(C) -2
(D) 0
(E) 9
(F) 18

boundary

$\vec\nabla f = \lambda \vec\nabla g$

$(2x-6, 2y-8) = (2\lambda x, 2\lambda y)$

$x-3 = \lambda x$, $y-4 = \lambda y$

$x = \frac{3}{1-\lambda}$, $y = \frac{4}{1-\lambda}$

$\left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{4}{1-\lambda}\right)^2 = 1 \leadsto \lambda = 6$ or -4

$(x,y) = (-3/5, -4/5)$ or $(3/5, 4/5)$

interior

$\vec\nabla f = \vec 0$

$\Rightarrow (x,y) = (3,4) \leftarrow$ but not in disk.

Now $f(-3/5, -4/5) > 0$ while $f(3/5, 4/5) = -2$.

(2) Let $f(x,y)$ be a function on the rectangle $R = [4,5] \times [2,4]$, with values given in the table:

	$x=4$	$x=4.5$	$x=5$
$y=2$	1	2	3
$y=3$	8	9	4
$y=4$	7	6	5

Taking $m = n = 2$ and assuming the maximum and minimum values of f on each subrectangle R_{ij} occur at a vertex, find the Riemann sum value which gives the best (i.e. least) upper bound on $\iint_R f(x,y)dA$.

(A) 36
(B) 30
(C) 24
(D) 18
(E) 15
(F) 12

Take $\sum_i \sum_j f(x_{ij}^*, y_{ij}^*) \underbrace{\Delta A}_{\Delta x \Delta y = 0.5 \cdot 1 = 0.5} = \frac{1}{2}(9+9+9+9) = \frac{36}{2} = 18$.

(3) Determine the volume of the solid under $z = 4xy + x^2$ over the rectangle $R = [1,2] \times [0,3]$.

(A) 34
(B) 41
(C) 46
(D) 51
(E) 61
(F) 68

$$\int_1^2\int_0^3 (4xy + x^2)\,dy\,dx = \int_1^2 \left[2xy^2 + x^2y\right]_{y=0}^3 dx$$

$$= \int_1^2 (18x + 3x^2)\,dx = \left[9x^2 + x^3\right]_1^2 = 9(4-1) + (8-1) = 27 + 7 = 34.$$

(4) Evaluate $\int_0^{\sqrt{\ln 2}} \int_0^1 \frac{xy\,e^{x^2}}{1+y^2}\,dy\,dx$.

(A) $\frac{1}{4}\ln 2$
(B) $\frac{1}{2}\ln 2$
(C) $\ln 2$
(D) $\frac{1}{4}$
(E) $\frac{1}{2}$
(F) 1

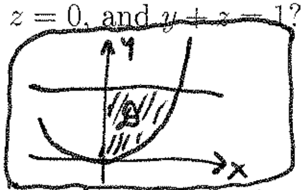
$$= \left(\int_0^{\sqrt{\ln 2}} x\,e^{x^2}\,dx\right)\left(\int_0^1 \frac{y}{1+y^2}\,dy\right)$$

$$= \left[\frac{1}{2}e^{x^2}\right]_0^{\sqrt{\ln 2}} \times \left[\frac{1}{2}\ln(1+y^2)\right]_0^1$$

$$= \left(1 - \frac{1}{2}\right) \times \left(\frac{1}{2}\ln 2 - 0\right) = \frac{1}{4}\ln 2$$

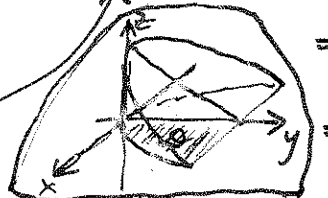
(5) What is the volume of the solid in the first octant bounded by the cylinder $y = x^2$ and the planes $x = 0$, $z = 0$, and $y + z = 1$?

(A) $\frac{64}{15}$
(B) $\frac{32}{15}$
(C) $\frac{16}{15}$
(D) $\frac{8}{15}$
(E) $\frac{4}{15}$
(F) $\frac{2}{15}$



Find volume under

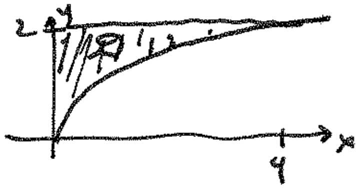
$y+z=1 \to z = 1-y$



$$\int_0^1\left(\int_{x^2}^1 (1-y)\,dy\right)dx = \int_0^1 \left[y-\frac{y^2}{2}\right]_{y=x^2}^1 dx$$
$$= \int_0^1\left(\frac12 - x^2 + \frac{x^4}{2}\right)dx$$
$$= \left[\frac12 x - \frac{x^3}{3}+\frac{x^5}{10}\right]_0^1 = \frac12-\frac13+\frac1{10}=\frac4{15}$$

(6) Compute $\int_0^4 \int_{\sqrt{x}}^2 \cos(y^3)\,dy\,dx$. ~~[Hint: you will~~ have to draw the region of integration to do this one.]

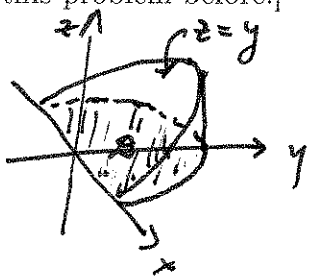
(A) $\frac13(1-\cos 8)$
(B) $\frac13 \sin 8$
(C) $\frac13 \cos 8$
(D) $3(1-\cos 8)$
(E) $3 \sin 8$
(F) $3 \cos 8$



$$= \iint_D \cos y^3\,dA = \int_0^2\int_0^{y^2}\cos y^3\,dx\,dy$$
$$= \int_0^2 y^2\cos y^3\,dy = \left(\frac13\sin y^3\right)\Big|_0^2$$
$$= \frac13\sin 8$$

(7) Find the volume of a wedge cut from a tall right solid circular cylinder of radius 5 (sitting on the xy plane) by a plane through a diameter of the base and making a 45° angle with the base. [Hint: you have done this problem before.]

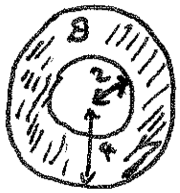
(A) $\frac13$
(B) $\frac53$
(C) $\frac{10}{3}$
(D) $\frac{50}{3}$
(E) $\frac{125}{3}$
(F) $\frac{250}{3}$



$$\iint_D y\,dA = \int_0^\pi\int_0^5 r\sin\theta\; r\,dr\,d\theta$$
$$= \left(\int_0^\pi \sin\theta\,d\theta\right)\left(\int_0^5 r^2\,dr\right)$$
$$= \left[-\cos\theta\right]_0^\pi\cdot\left[\frac{r^3}{3}\right]_0^5 = 2\cdot\frac{5^3}{3} = \frac{250}{3}$$

(8) Evaluate $\iint_D \frac{1}{x^2+y^2}\,dA$, where D is the region between the circles $x^2+y^2=4$ and $x^2+y^2=16$.

(A) $\frac{\pi}{2}$
(B) π
(C) 2π
(D) $\pi\ln 2$
(E) $2\pi\ln 2$
(F) $2\pi\ln 4$



$$\int_0^{2\pi}\int_2^4 \frac{1}{r^2}\,r\,dr\,d\theta = 2\pi\int_2^4\frac{dr}{r}$$
$$= 2\pi(\ln 4-\ln 2) = 2\pi\ln 2$$

(9) Determine the center of mass of the homogeneous (i.e. of constant density) lamina bounded by the cardioid $r = 1+\sin\theta$. [Hint: you may want to use $2\sin^2\theta = 1-\cos^2 2\theta$ and $2\cos\theta\sin\theta = \sin 2\theta$.]

(A) $(0, \frac{10}{9})$
(B) $(0, \frac56)$
(C) $(0, 1)$
(D) $(0, \frac23)$
(E) $(0, \frac{8}{15})$
(F) $(0, \frac{16}{15})$

$$M_x = \iint_D y\,dA = \int_0^{2\pi}\int_0^{1+\sin\theta} r\sin\theta\; r\,dr\,d\theta = \int_0^{2\pi}\sin\theta\left[\frac{r^3}{3}\right]_{r=0}^{1+\sin\theta}d\theta$$
$$= \frac13\int_0^{2\pi}\sin\theta(1+\sin\theta)^3\,d\theta = \frac13\int_0^{2\pi}(\sin\theta+3\sin^2\theta+3\sin^3\theta+\sin^4\theta)\,d\theta$$

throw out odd powers, use $\sin^4\theta = \sin^2\theta - \sin^2\cos^2 = \sin^2 - \frac14\sin^2 2\theta$

$$= \frac13\int_0^{2\pi}\left(4\sin^2\theta - \frac14\sin^2 2\theta\right)$$
$$= \frac13\int_0^{2\pi}\left(2-2\cos 2\theta - \frac{1-\cos 4\theta}{8}\right)d\theta$$

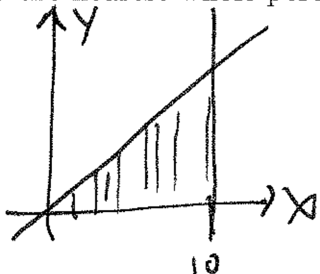
again, throw out

$$= \frac13\int_0^{2\pi}\frac{15}{8}\,d\theta = \frac13\cdot\frac{15}{8}\cdot 2\pi = \frac{5\pi}{4}.$$

Similarly, $m = \iint_D dA = \int_0^{2\pi}\int_0^{1+\sin\theta} r\,dr\,d\theta = \dots = \frac{3\pi}{2}$. So $\bar y = \frac{M_x}{m} = \frac{5\pi/4}{3\pi/2} = \frac56$.

(10) Let X [resp. Y] denote the number of inches of snow that will fall in St. Louis [resp. Boston] as independent weather systems move through each city. Assume X has probability distribution $F(x) = \begin{cases} \frac{1}{10}, & 0 \leq x \leq 10 \\ 0, & \text{otherwise} \end{cases}$ and Y has probability distribution $G(y) = \begin{cases} \frac{1}{10}e^{-y/10}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$. Calculate the probability that St. Louis gets more snow than Boston, and then decide which approximation (to the nearest whole percentage) is correct:

(A) 1%
(B) 17%
(C) 37%
(D) 51%
(E) 73%
(F) 99%



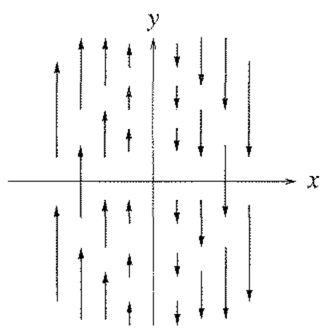
$$\int_0^{10}\int_0^x \frac{1}{10}\cdot\frac{1}{10}e^{-y/10}\,dy\,dx = \frac{1}{100}\int_0^{10}\left(\int_0^x e^{-y/10}\,dy\right)dx$$
$$= \frac{1}{10}\int_0^{10}(1-e^{-x/10})\,dx = \frac{1}{10}\left[x+10e^{-x/10}\right]_0^{10}$$
$$= \frac{1}{10}\left(10+10e^{-10/10}-10\right) = \frac{1}{e} \approx 37\%$$
(i.e. 0.37)

(11) If we wish to change variables in a double integral from (x, y) to (u, v), where $x = uv$ and $y = u/v$, we must replace $dx\,dy$ by what function times $du\,dv$?

(A) 2
(B) -2
(C) u^2
(D) $-u/v$
(E) $2v/u$
(F) $-2u/v$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}.$$

(12) Which vector field is sketched in the picture?



(A) $-y\hat{\imath} + x\hat{\jmath}$
(B) $x\hat{\imath}$
(C) $-y\hat{\imath}$
(D) $-x\hat{\jmath}$
(E) $x\hat{\imath} + y\hat{\jmath}$
(F) $y\hat{\jmath}$

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PART II: HAND-GRADED PROBLEMS

This part has two pages. Show all the work you want graded for each problem in the space provided. Please **print your name at the top of each page.**

(1) [16 points] Maximize and minimize $f(x, y, z) = yz + xy$ subject to the two constraints $xy = 1$, $y^2 = 1 - z^2$.

Take $g = xy$, $h = y^2 + z^2$

$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$

$\langle y, x+z, y \rangle = \lambda \langle y, x, 0 \rangle + \mu \langle 0, 2y, 2z \rangle$

$y = \lambda y$, $x + z = \lambda x + 2\mu y$, $y = 2\mu z$

$\Downarrow$

$(\cancel{y=0})$ violates $xy = 1$ or $\lambda = 1 \Rightarrow x + z = x + 2\mu y$

$\Downarrow$

$z = 2\mu y \longrightarrow z = 4\mu^2 z$

$\Downarrow$

$z = 0$ or $\mu = \pm \frac{1}{2}$.

- If $z = 0$, then $y = \pm 1 \xRightarrow{xy=1} (x, y) = (1,1)$ or $(-1,-1)$.

  $f(1,1,0) = 1$, $f(-1,-1,0) = 1$.

- If $\mu = \pm \frac{1}{2}$, then $z = \pm y \Rightarrow y^2 = 1 - y^2 \Rightarrow y = \pm \frac{1}{\sqrt{2}} \xRightarrow{xy=1} x = \pm\sqrt{2}$

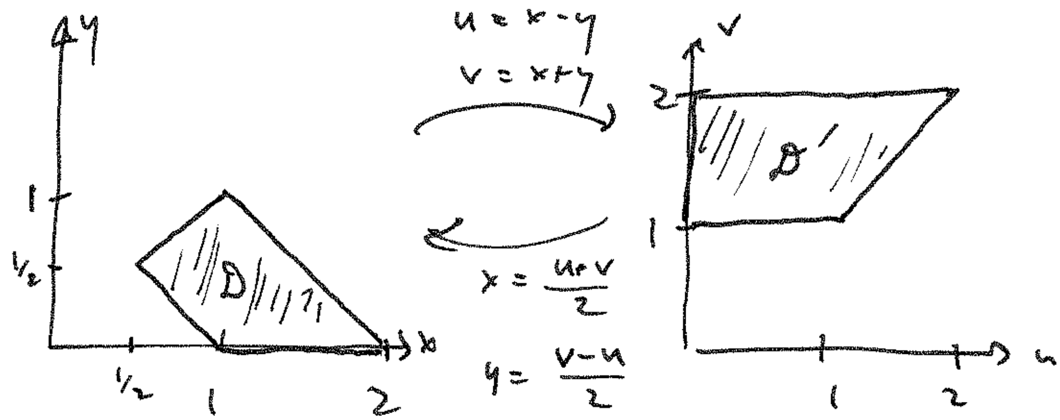
  $\left.\begin{array}{l} f(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1 + \frac{1}{2} = \frac{3}{2} \\ f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1 + \frac{1}{2} = \frac{3}{2} \end{array}\right\}$ max $= \boxed{3/2}$

  $\left.\begin{array}{l} f(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1 - \frac{1}{2} = \frac{1}{2} \\ f(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1 - \frac{1}{2} = \frac{1}{2} \end{array}\right\}$ min $= \boxed{1/2}$

(2) [4 points] What does it mean for a vector field $\vec{F}$ to be conservative? Is $\vec{F}(x, y) = x\hat{i}$ conservative?

- $\boxed{\text{It means that } \vec{F} = \vec{\nabla} f \text{ for some } f}$.
- $\boxed{\text{Yes}}$: $x\hat{i} = \vec{\nabla}\left(\frac{x^2}{2}\right)$

(3) [20 points] Evaluate $\iint_D \exp\left(\frac{x-y}{x+y}\right) dA$, where D is the quadrilateral with vertices at $(1,0)$, $(2,0)$, $(1,1)$, $(\frac{1}{2},\frac{1}{2})$, by making an appropriate change of variables. [Remark: your answer should be positive.]



$$dx\,dy = \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv, \quad \text{where} \quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{2}.$$

$$\Rightarrow \iint_D e^{\frac{x-y}{x+y}}\,dA = \iint_{D'} e^{u/v}\,\tfrac{1}{2}\,dA = \frac{1}{2}\int_1^2\left(\int_0^v e^{u/v}\,du\right)dv$$

$$= \frac{1}{2}\int_1^2 \left[v e^{u/v}\right]_{u=0}^{v} dv = \frac{e-1}{2}\int_1^2 v\,dv$$

$$= \frac{e-1}{2}\left[\frac{v^2}{2}\right]_1^2 = \frac{e-1}{2}\left(\frac{4}{2}-\frac{1}{2}\right) = \boxed{\frac{3(e-1)}{4}}.$$