MATH 233 MIDTERM EXAM 1   SOLUTIONS

This exam consists of 20 multiple choice (machine-graded) problems, worth 5 points each (for a total of 100 points). No 3x5 cards or calculators are allowed. You will need a **pencil** to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit, and **shade in the corresponding boxes.** Also **print your name at the top of your card.**

grading scale

100	A+
85-95	A
80	A−
75	B+
70	B
65	B−
60	C+
55	C
50	C−
35-45	D
0-30	F

(1) Let $\vec{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $\vec{b} = 5\mathbf{i} + \mathbf{j} + \mathbf{k}$. Compute $\vec{a} \cdot \vec{b}$.
(A) 0
(B) 3
(C) 6
(D) 9
(E) 12
(F) 15

$$\langle 2,3,-4\rangle \cdot \langle 5,1,1\rangle = 10+3-4=9$$

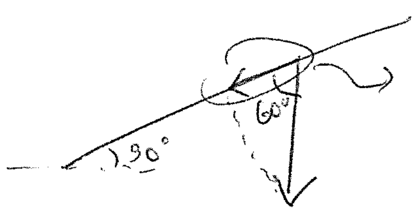
(2) With $\vec{a}, \vec{b}$ as in problem (1), find $\vec{a} \times \vec{b}$.
(A) $-13\mathbf{k} - 22\mathbf{j} + 7\mathbf{k}$
(B) $-\mathbf{i} - 11\mathbf{j} + 22\mathbf{k}$
(C) $7\mathbf{i} + 22\mathbf{j} - 13\mathbf{k}$
(D) $13\mathbf{i} + 22\mathbf{j} + 7\mathbf{k}$
(E) $\mathbf{i} - 22\mathbf{j} + 13\mathbf{k}$
(F) $7\mathbf{i} - 22\mathbf{j} - 13\mathbf{k}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -4 \\ 5 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 1 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 2 & -4 \\ 5 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}\hat{k}$$
$$= 7\hat{i} - 22\hat{j} + 13\hat{k}$$

(3) With $\vec{a}, \vec{b}$ as in problem (1), compute the vector projection $\mathbf{proj}_{\vec{b}}\vec{a}$ of $\vec{a}$ onto $\vec{b}$.
(A) $7\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$
(B) $\frac{5}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$
(C) $\frac{45}{29}\mathbf{i} + \frac{9}{29}\mathbf{j} + \frac{9}{29}\mathbf{k}$
(D) $\frac{18}{29}\mathbf{i} + \frac{27}{29}\mathbf{j} - \frac{36}{29}\mathbf{k}$
(E) $\frac{10}{3}\mathbf{i} + 5\mathbf{j} - \frac{20}{3}\mathbf{k}$
(F) $\frac{2}{3}\mathbf{i} + \mathbf{j} - \frac{4}{3}\mathbf{k}$

$$\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b} = \frac{9}{27}\langle 5,1,1\rangle = \langle \tfrac{5}{3}, \tfrac{1}{3}, \tfrac{1}{3}\rangle$$

(4) A 40 kg package sits on a frictionless 30° incline. How much force (in Newtons), applied parallel to the surface of the incline, will just prevent the package from sliding down? (Use $g = 10\,m/s^2$ for the acceleration due to gravity in this and the next problem.)
(A) 100
(B) 200
(C) 300
(D) 400
(E) 500
(F) 600

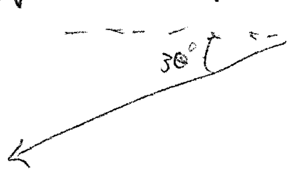


component of gravity along surface

$$= \tfrac{1}{2}\cdot mg = \tfrac{1}{2}\cdot 40 \cdot 10 = 200$$

(5) If the surface of the incline in problem (4) is 20 m long, and the 40 kg package is allowed to slide from the top to the bottom, how much work (in Newton-meters) does gravity do in the process?
(A) 1000
(B) 2000
(C) 3000
(D) 4000
(E) 5000
(F) 6000

$$W = \vec{F}\cdot\vec{d} = \langle 0, -mg\rangle \cdot \langle -20\cos 30°, -20\sin 30°\rangle$$
$$= \langle 0, -400\rangle \cdot \langle -20\tfrac{\sqrt{3}}{2}, -10\rangle$$
$$= 4000$$



(6) Find the area of the triangle with vertices $(0,0,0)$, $(1,2,-2)$, and $(-1,0,1)$.
(A) $\frac{1}{2}$
(B) 1
(C) $\frac{3}{2}$
(D) 2
(E) $\frac{5}{2}$
(F) 3

$A = \frac{1}{2}\|\langle 1,2,-2\rangle \times \langle -1,0,1\rangle\| = \frac{1}{2}\|\langle 2,1,2\rangle\|$
$= \frac{1}{2}\sqrt{2^2+1^2+2^2} = 3/2$

(7) Which plane is parallel to the one containing the triangle of problem (6)?
(A) $-4x-2y-4z=6$
(B) $4x-2y+4z=6$
(C) $2x+y-2z+7=0$
(D) $2x+y+z+9=0$
(E) $x+2y-2z=1$
(F) $2x-2z=9$

normal vector must be a multiple of $\langle 2,1,2\rangle$

(8) Find the center and radius of the sphere $3x^2+3y^2+3z^2=49+6x-12y$.
(A) $(-1,2,0)$, $\frac{8}{3}$
(B) $(1,-2,0)$, $\frac{7}{\sqrt{3}}$
(C) $(1,2,0)$, 8
(D) $(-1,2,0)$, 7
(E) $(1,-2,0)$, $\frac{8}{\sqrt{3}}$
(F) $(1,2,0)$, $\frac{7}{3}$

complete square to get
$3(x-1)^2+3(y+2)^2+3z^2=64$
$(x-1)^2+(y+2)^2+z^2=64/3=(8/\sqrt{3})^2.$

(9) Determine the distance between the point $(-2,1,3)$ and the plane $3x+2y+6z+7=0$.
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

$$\text{dist} = \frac{|3(-2)+2(1)+6(3)+7|}{\sqrt{9+4+36}} = \frac{21}{7} = 3$$

(10) Which is an equation for the plane parallel to the plane of problem (9) and through $(-1,1,1)$?
(A) $3x+2y+6z=0$
(B) $-x-y-z+7=0$
(C) $-3x-2y-6z=10$
(D) $-3x+2y+6z=11$
(E) $6x+4y+12z=10$
(F) $x+y+z=1$

normal vector must be a multiple of $\langle 3,2,6\rangle$, and $(-1,1,1)=(x,y,z)$ must satisfy the equation.

(11) Let C be the curve with parametric equations $x=15\cos t$, $y=\sin t$. By eliminating the parameter to get a Cartesian equation, decide which one of the following points lies on C:
(A) $(15\sqrt{3}, \frac{1}{2})$
(B) $(\frac{15\sqrt{2}}{2}, \frac{15\sqrt{2}}{2})$
(C) $(15,-1)$
(D) $(9,-\frac{1}{2})$
(E) $(9,-\frac{4}{5})$
(F) $(-9,\frac{\sqrt{3}}{2})$

$\left(\frac{x}{15}\right)^2+y^2=\cos^2 t+\sin^2 t=1$

now plug in points...

(12) Find the angle between the planes $x + 2y - 2z = 3$ and $x - z = 1$.

(A) they are parallel
(B) 30°
(C) 45°
(D) 60°
(E) 90°
(F) none of the above

$\angle$ = angle θ between their normal vectors $\vec{n}_1$ & $\vec{n}_2$

$$\cos\theta = \frac{\langle 1,2,-2\rangle\cdot\langle 1,0,-1\rangle}{\|\langle 1,2,-2\rangle\|\,\|\langle 1,0,-1\rangle\|} = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$\Longrightarrow \theta = 45°$

(13) What are equations for the line at the intersection of the two planes in problem (12)?

(A) $x+1 = \frac{y-1}{2} = -z$
(B) $\frac{x-1}{2} = y+1 = \frac{z}{-2}$
(C) $x-1 = \frac{y+1}{2} = \frac{z}{2}$
(D) $\frac{x-1}{2} = \frac{y-1}{2} = z$
(E) $x-1 = \frac{y-1}{2} = z$
(F) $\frac{x-1}{2} = y-1 = \frac{z}{2}$

$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \langle 1,2,-2\rangle \times \langle 1,0,-1\rangle = \langle -2,-1,-2\rangle$

direction vector

(can also use $\langle 2,1,2\rangle$) or any other multiple of $\vec{v}$

↳ only (F) satisfies this.

(14) Find the distance between the lines $\frac{x-3}{2} = \frac{y+2}{-2} = z-1$ and $x-4 = \frac{y+6}{2} = \frac{z}{2}$.

(A) 0: they intersect
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

write parametric equations $(x,y,z) = (3+2t, -2-2t, 1+t)$ for first line, plug into second line's equations:

$$3+2t-4 = \frac{-2-2t+6}{2} = \frac{1+t}{2}$$

↳ $2t-1 = 2-t = \frac{1+t}{2}$ → solution $t=1$ ⇒ intersect at $(5,-4,2)$.

(15) Describe the quadric surface consisting of points in $\mathbb{R}^3$ whose distance to $P(1,0,0)$ is twice their distance to the z-axis.

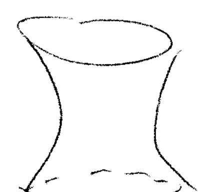
(A) elliptic paraboloid
(B) hyperboloid of 1 sheet
(C) ellipsoid
(D) hyperboloid of 2 sheets
(E) hyperbolic paraboloid
(F) hyperactive zebroid

$d(P,Q) = 2\,d(z\text{-axis}, Q)$ $\quad (Q(x,y,z))$

$\sqrt{(x-1)^2+y^2+z^2} = 2\sqrt{x^2+y^2}$

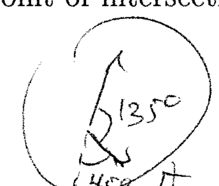
$x^2-2x+1+y^2+z^2 = 4x^2+4y^2$

$\frac{4}{3} = 3(x+\frac{1}{3})^2 + 3y^2 - z^2$



(16) Let $f(x) := x^3$ and $g(x) := x^2 - 2$. Find the angle between the curves $y = f(x)$ and $y = g(x)$ at their (unique) point of intersection.

(A) 0°
(B) 30°
(C) 45°
(D) 60°
(E) 90°
(F) none of the above



↳ $(x,y) = (-1,-1)$.

$f'(x) = 3x^2$, $g'(x) = 2x$ ⇝ tangent vectors are $\langle 1,3\rangle$ and $\langle 1,-2\rangle$.

$$\Rightarrow \cos\theta = \frac{\langle 1,3\rangle\cdot\langle 1,-2\rangle}{\sqrt{10}\,\sqrt{5}} = \frac{-5}{5\sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow \theta = 135°$$

(17) The flight path of a bug is described parametrically by $x = \frac{1}{3}t^3$, $y = 1 + \frac{1}{2}t^2$, $z = -t$ (for $t \geq 0$). At what time t is the bug flying parallel to the window with equation $2x + 7y + 15z = 10$?

(A) at all times
(B) $\frac{1}{2}$
(C) 1
(D) $\frac{3}{2}$
(E) 2
(F) $\frac{5}{2}$

i.e. when is $\vec{r}\,'(t) \perp \vec{n}$ (= normal vector)

$$0 = \vec{n}\cdot\vec{r}\,'(t) = \langle 2,7,15\rangle\cdot\langle t^2, t, -1\rangle$$
$$= 2t^2+7t-15 = 2(t+5)(t-3/2)$$

ignore $t=-5$ since <0.

(18) Identify $r^2 \sin 2\theta = 2r\cos\theta + 1$ as a conic.

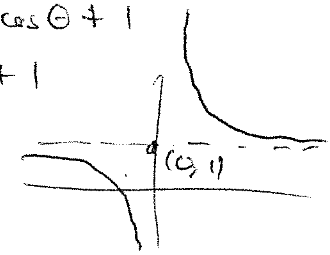
$= 2\sin\theta\cos\theta$

(A) circle with center $(1, 0)$
(B) ellipse with center $(0, 1)$
(C) parabola with vertex at $(1, 0)$
(D) hyperbola centered at $(0, 1)$
(E) ellipse with center $(0, -1)$
(F) hyperbola centered at $(-1, 0)$

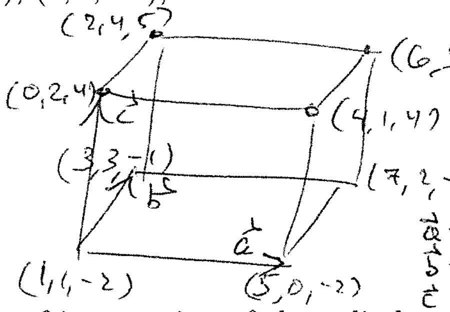
$2r\cos\theta\, r\sin\theta = 2r\cos\theta + 1$

$2xy = 2x + 1$

$2x(y-1) = 1$



(19) The basement floor of a parallelepiped-shaped art museum has corners with coordinates $(1, 1, -2)$, $(5, 0, -2)$, $(3, 3, -1)$, $(7, 2, -1)$, and roof with coordinates $(0, 2, 4)$, $(4, 1, 4)$, $(2, 4, 5)$, $(6, 3, 5)$. What is its volume?
(A) 51
(B) 54
(C) 57
(D) 60
(E) 63
(F) 66



$V = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 2 & 1 \\ -1 & 1 & 6 \end{vmatrix} = 4\begin{vmatrix} 2 & 1 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ -1 & 6 \end{vmatrix}$

$\vec{a} = \langle 4, -1, 0 \rangle$
$\vec{b} = \langle 2, 2, 1 \rangle$
$\vec{c} = \langle -1, 1, 6 \rangle$

$= 44 + 13 = 57$

(20) Let C be the curve of intersection of the cylinder $x^2 + z^2 = 2$ and the surface $y = xz$. Let ℓ be the tangent line to C at the point $(1, 1, 1)$. Which one of the following points lies on this line? [Hint: first, find the vector/parametric equation for C.]
(A) $(-1, 3, 1)$
(B) $(4, -1, 0)$
(C) $(2, 1, -1)$
(D) $(-3, 1, 2)$
(E) $(3, 2, -1)$
(F) $(-2, 1, 4)$

parametrize C by

$\vec{r}(t) = \langle \underbrace{\sqrt{2}\cos t}_{x},\ \underbrace{2\cos t\sin t}_{y(=xz)},\ \underbrace{\sqrt{2}\sin t}_{z} \rangle$

$= \langle \sqrt{2}\cos t,\ \sin 2t,\ \sqrt{2}\sin t \rangle$

at $t = \pi/4$, this gives $\langle \sqrt{2}\frac{\sqrt{2}}{2}, 1, \sqrt{2}\frac{\sqrt{2}}{2} \rangle = \langle 1, 1, 1 \rangle$.

$\vec{r}\,'(\pi/4) = \langle -\sqrt{2}\sin\pi/4,\ 2\cos(2\cdot\frac{\pi}{4}),\ \sqrt{2}\cos\frac{\pi}{4} \rangle$

$= \langle -1, 0, 1 \rangle$

$\rightsquigarrow \vec{\ell}_{\pi/4}(u) = \langle 1 - u, 1, 1+u \rangle$

only (D) & (F) have $y = 1$, and it's easy to see (D) isn't on the line and (F) is.