MATH 233 MIDTERM EXAM 2 SOLUTIONS

This exam consists of 20 multiple choice (machine-graded) problems, worth 5 points each (for a total of 100 points). No 3x5 cards or calculators are allowed. You will need a **pencil** to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit, and **shade in the corresponding boxes.** Also **print your name at the top of your card.**

Grading scale

95-100	A+	50	B-
75-90	A	45	C+
70	A-	40	C
65	B+	30-35	D
55-60	B	below 30	F

(1) If $F(x,y) = \ln(x^2+xy+y^2)$, what is $F_x(2,-1)$?
- (A) 6
- (B) 5
- (C) 4
- (D) 3
- (E) 2
- [✓ (F) 1]

$$F_x = \frac{2x+y}{x^2+xy+y^2}$$

$$F_x(2,-1) = \frac{4-1}{4-2+1} = \frac{3}{3} = 1$$

(2) The level curves of $f(x,y) = \frac{e^{x^2}}{e^{y^2}}$ are ...
- (A) bell curves
- (B) ellipses
- [✓ (C) hyperbolas]
- (D) parabolas
- (E) like the graph of ln
- (F) lines

Set $k = f(x,y) = e^{x^2-y^2}$

$$\underbrace{\ln k}_{\text{Constant}} = x^2 - y^2$$

(3) Find $\vec{r}(\frac{\pi}{2})$ if $\vec{a}(t) = \langle -\cos t, \sin t\rangle$, $\vec{v}(0) = \langle 1,0\rangle$, and $\vec{r}(0) = \langle 1,3\rangle$.
- (A) $\langle -\frac{\pi}{2}, 2-\frac{\pi}{2}\rangle$
- (B) $\langle \pi, 2+\pi\rangle$
- [✓ (C) $\langle \frac{\pi}{2}, 2+\frac{\pi}{2}\rangle$]
- (D) $\langle \pi, 2-\pi\rangle$
- (E) $\langle -\frac{\pi}{2}, 2+\frac{\pi}{2}\rangle$
- (F) $\langle -\pi, 2+\pi\rangle$

$$\vec{v} = \int \vec{a}\,dt = \langle -\sin t, -\cos t\rangle + \vec{C}$$

$$\langle 1,0\rangle = \vec{v}(0) = \langle 0,-1\rangle + \vec{C} \Rightarrow \vec{C} = \langle 1,1\rangle$$

$$\vec{r} = \int \vec{v}\,dt = \langle t+\cos t, t-\sin t\rangle + \vec{K}$$

$$\langle 1,3\rangle = \vec{r}(0) = \langle 1, 0\rangle + \vec{K} \Rightarrow \vec{K} = \langle 0,3\rangle$$

So $\vec{r} = \langle t+\cos t, 3+t-\sin t\rangle$

and $\vec{r}(\pi/2) = \langle \frac{\pi}{2}, 2+\frac{\pi}{2}\rangle$

(4) After you are picked up by your Uber driver in St. Louis, your path of motion is described by $\vec{r}(t) = \langle 2\sin t, 2\cos t, \sqrt{5}t\rangle$ (in meters) from $t = 0$ to $t = 10$ (seconds). Calculate the length (in meters) of the arc along which you traveled (in a parking lot or tornado, as the case may be).

(A) 15
[✓ (B) 30]
(C) 45
(D) 60
(E) 75
(F) 90

$$\vec{r}\,'(t) = \langle 2\cos t, -2\sin t, \sqrt{5}\rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{\underbrace{4\cos^2 t + 4\sin^2 t}_{4} + 5} = \sqrt{9} = 3$$

$$\underline{\text{arclength}} = \int_0^{10} \|\vec{r}\,'(t)\|\,dt = \int_0^{10} 3\,dt = 30$$

(5) If the temperature distribution $T(x, y, z)$ has $T_x(0, 2, 0) = 1$, $T_y(0, 2, 0) = -1$, and $T_z(0, 2, 0) = -\sqrt{5}$ (in degrees Celsius), how fast (in degrees/second) is the outside temperature changing along your path in problem (4) at time $t = 0$?

[✓ (A) −3]
(B) −2
(C) −1
(D) 0
(E) 1
(F) 2

<u>use Chain rule:</u>

first, $\vec{r}(0) = \langle 0, 2, 0\rangle$ and

$x(t) = 2\sin t$, $y(t) = 2\cos t$, $z(t) = \sqrt{5}t$

$x'(0) = 2\cos 0 = 2$, $y'(0) = -2\sin 0 = 0$, $z'(0) = \sqrt{5}$

$$\left.\frac{dT}{dt}\right|_{t=0} = T_x(0,2,0)\,x'(0) + T_y(0,2,0)\,y'(0) + T_z(0,2,0)\,z'(0)$$

$$= 1\cdot 2 + (-1)\cdot 0 + (-\sqrt{5})\sqrt{5}$$

$$= 2 - 5 = -3$$

(6) Which of the following could be $f_x(x, y)$ if $f_y(x, y) = 2x^3y - y^3$?

(A) $2 - y^3 + 3x^2y^2$
(B) $\frac{1}{2}x^4 - 3y^2$
(C) $x^2y^2 - 3y^2 + 1$
(D) $e^x - 6 + x^2y^2$
[✓ (E) $5 + x^3 + 3x^2y^2$]
(F) $x^4 + 2$

<u>use Clairaut's theorem:</u>

$$(f_x)_y = (f_y)_x = \frac{\partial}{\partial x}(2x^3y - y^3) = 6x^2y$$

taking antiderivative with respect to y, we get

$$f_x = 3x^2y^2 + (\text{any function of } \underline{\underline{x}})$$

(7) Suppose a tree trunk has a radius of 10 inches, currently increasing at $\frac{1}{2}$ inch per year, and a height of 200 inches, currently increasing at 4 inches per year. (Assume the tree trunk is a right circular cylinder.) How fast is the volume of the tree trunk currently increasing, in cubic inches per year?

(A) 2000π
[✓ (B) 2400π]
(C) 2800π
(D) 3200π
(E) 3600π
(F) 4000π

<u>use Chain rule:</u>

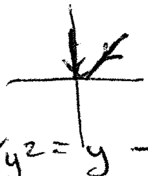
$$V = \pi r^2 h$$

$$\frac{dV}{dt} = V_r\, r'(t) + V_h\, h'(t)$$
$$= (2\pi r h)\cdot r'(t) + (\pi r^2)\cdot h'(t)$$
$$= 2\pi(10)(200)\cdot\frac{1}{2} + \pi(10)^2\cdot 4$$
$$= 2400\pi$$

substitute in info

(8) Which of the following limits exist: (a) $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$; (b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$; (c) $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$?

(A) (a) only
(B) (b) only
(C) (c) only
(D) (a) and (c)
(E) (a) and (b)
(F) (b) and (c)

(a) DNE: e.g. consider  as your paths

$x=0$ gives $y^3/y^2 = y \longrightarrow 0$

$x=y$ gives $\frac{x^2+x^3}{2x^2} = \frac{1}{2}+\frac{1}{2}x \longrightarrow \frac{1}{2}$

(b) DNE: e.g. use polar form $\frac{\cancel{r^2}\boxed{\cos\theta\sin\theta}}{\cancel{r^2}}$

(c) is 0: use $0 \le \frac{y^2}{x^2+y^2} \le 1$

$$\underbrace{-|x|}_{\searrow} \le \frac{xy^2}{x^2+y^2} \le \underbrace{|x|}_{\swarrow} \quad \text{(squeeze lemma)}$$
$$\overset{\wr}{0}$$

(9) Find the equation for the tangent plane to $z = xe^{-2y}$ at $(1,0,1)$.

(A) $1 = x - 2y + z$
(B) $z = x$
(C) $z = x + 2y$
(D) $1 = x + 2y + z$
(E) $z = x - 2$
[✓ (F) $z = x - 2y$]

let $f(x,y) = xe^{-2y}$. Use linear approximation:

$f_x = e^{-2y}$, $f_y = -2xe^{-2y}$

$f_x(1,0) = 1$, $f_y(1,0) = -2$

$$\underline{z = L(x,y)} = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0)$$
$$= x-1 + (-2)y + 1$$
$$= x - 2y$$

tangent plane

(10) Find the minimum distance between the point $(1, 2, 0)$ and the quadric cone $z^2 = x^2 + 7y^2$.

(A) 1
[✓ (B) 2]
(C) 3
(D) 4
(E) 5
(F) 6

$$(\text{distance})^2 = (x-1)^2 + (y-2)^2 + z^2$$
$$= (x-1)^2 + (y-2)^2 + x^2 + 7y^2$$
$$= 2x^2 + 8y^2 - 2x - 4y + 5 =: F(x,y)$$

Set $\left.\begin{cases} 0 = F_x = 4x - 2 \Rightarrow x = \frac{1}{2} \\ 0 = F_y = 16y - 4 \Rightarrow y = \frac{1}{4} \end{cases}\right\} \Rightarrow \text{distance} = \sqrt{F(\tfrac{1}{2},\tfrac{1}{4})}$

$= \sqrt{4} = 2$.

(11) With $f(x,y) = x^2y$, find the slope of the graph of $z = f(x,y)$ in the direction $\vec{v} = \langle 3, -4\rangle$, "over" the point $P(1,2)$.

(A) 1
(B) $\frac{4}{5}$
(C) 8
(D) $\frac{1}{5}$
[✓ (E) $\frac{8}{5}$]
(F) 4

use <u>directional derivative</u>: first,

$$\hat{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 3,-4\rangle}{\sqrt{3^2+4^2}} = \langle \tfrac{3}{5}, \tfrac{-4}{5}\rangle$$

$$\vec{\nabla} f = \langle 2xy, x^2\rangle \Rightarrow (\vec{\nabla} f)(1,2) = \langle 4, 1\rangle$$

$$(D_{\hat{u}} f)(1,2) = (\vec{\nabla} f)(1,2) \cdot \langle \tfrac{3}{5}, \tfrac{-4}{5}\rangle = \frac{8}{5}.$$

(12) With f and P as in problem (11), which vector is tangent to the level curve of $f(x,y)$ through P?

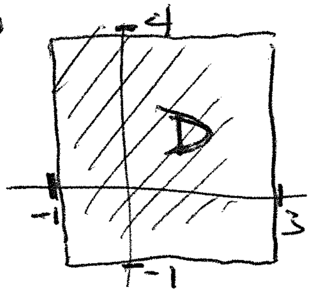
[✓ (A) $\langle 1, -4\rangle$]
(B) $\langle 4, 1\rangle$
(C) $\langle -4, 1\rangle$
(D) $\langle 3, 4\rangle$
(E) $\langle 4, -3\rangle$
(F) $\langle 1, -3\rangle$

$(\vec{\nabla} f)(1,2)$ is $\perp$ to the level curve

So you need something $\perp$ to $\langle 4, 1\rangle$

(13) Which of the following is the (absolute) maximum value of $f(x,y) = x^2 - y^2 + 1$ on the set $D = \{(x,y) \mid -1 \le x \le 3,\ -1 \le y \le 4\}$?

(A) 1
(B) 5
(C) 9
[✓ (D) 10]
(E) 14
(F) 26



boundary: $f(-1,y) = 2-y^2$ max $= 2$
$f(3,y) = 10-y^2$ max $= 10$
both at $y=0$
$f(x,-1) = x^2$ max $= 9$
$f(x,4) = x^2-15$ max $= -6$
both at $x=3$

interior: $\left.\begin{array}{l} 0 = f_x = 2x \\ 0 = f_y = 2y \end{array}\right\} \Rightarrow (x,y) = (0,0)$
$f(0,0) = \boxed{1}$

Since the maximum must be among these values, it is 10.

(14) The radius of the osculating circle to $\vec{r}(t) = \langle 2t, \cos t, \sin t\rangle$ is everywhere equal to:

(A) $\frac{1}{5}$
(B) 1
[✓ (C) 5]
(D) $\frac{5}{2}$
(E) $\frac{4}{5}$
(F) $\frac{2}{5}$

$\vec{r}\,' = \langle 2, -\sin t, \cos t\rangle \Rightarrow \|\vec{r}\,'\| = \sqrt{5}$
$\vec{r}\,'' = \langle 0, -\cos t, -\sin t\rangle$
$\vec{r}\,' \times \vec{r}\,'' = \langle 1, 2\sin t, -2\cos t\rangle \Rightarrow \|\vec{r}\,' \times \vec{r}\,''\| = \sqrt{5}$

$$\kappa = \frac{\|\vec{r}\,' \times \vec{r}\,''\|}{\|\vec{r}\,'\|^3} = \frac{\sqrt{5}}{(\sqrt{5})^3} = \frac{1}{5} \Rightarrow R = 5$$

(15) The osculating plane (for $\vec{r}$ as in problem (14)) at $t = \frac{\pi}{2}$ has equation:

(A) $x + 2z = \pi + 2$
(B) $x - 2y = \pi$
(C) $x + 2y = \pi + 2$
(D) $x - 2z = \pi - 2$
[✓ (E) $x + 2y = \pi$]
(F) $x - 2y = \pi - 2$

$\vec{r}(\pi/2) = \langle \pi, 0, 1\rangle$ ← plane contains this point

$\vec{B}$ is a multiple of $\vec{r}\,' \times \vec{r}\,''$, normal to osculating plane.

At $\pi/2$, get

$\vec{B}(\pi/2) =$ mult. of $\langle 1, 2, 0\rangle$

$$\Rightarrow 0 = 1(x-\pi) + 2(y-0) + 0(z-1)$$
$$= x + 2y - \pi$$

$\Rightarrow$ eqn is $\pi = x + 2y$

(16) How many of each type of critical point does $f(x,y) = 2x^4 - x^2 + 3y^2$ have: saddle point; local maximum; local minimum?

(A) 1; 2; 0
(B) 1; 1; 1
(C) 0; 0; 2
[✓ (D) 1; 0; 2]
(E) 0; 2; 0
(F) 0; 1; 1

$$0 = f_x = 8x^3 - 2x \Rightarrow x = 0, \pm\frac{1}{2}$$

$$0 = f_y = 6y \Rightarrow y = 0.$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 24x^2-2 & 0 \\ 0 & 6 \end{vmatrix} = 12(12x^2-1)$$

$$D(0,0) = -12 < 0 \Rightarrow \text{saddle}$$

$$\left\{\begin{array}{l} D(\tfrac{1}{2},0) = D(-\tfrac{1}{2},0) = 24 > 0 \\ f_{xx}(\tfrac{1}{2},0) = f_{xx}(-\tfrac{1}{2},0) = 4 > 0 \end{array}\right. \Rightarrow \text{2 local minima}$$

(17) The formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ determines the combined resistance R when resistors of resistance R_1 and R_2 are connected in parallel. Suppose that R_1 and R_2 were measured at 25 and 100 ohms, respectively, so that you calculate $R = 20$. If the possible errors in each of your two measurements were ±0.5 ohms each, use differentials to estimate the maximum possible error (in ohms) in the computed value of R.

(A) 0.16
(B) 0.22
(C) 0.28
[✓ (D) 0.34]
(E) 0.40
(F) 0.46

$$-\frac{1}{R^2}dR = -\frac{1}{R_1^2}dR_1 - \frac{1}{R_2^2}dR_2$$

$$\delta R = \frac{R^2}{R_1^2}dR_1 + \frac{R^2}{R_2^2}dR_2 = \left(\overset{4/5}{\cancel{\frac{20}{25}}}\right)^2\cdot 0.5 + \left(\overset{1/5}{\cancel{\frac{20}{100}}}\right)^2\cdot 0.5$$

$$= \frac{16+1}{25}\cdot(0.5) = \frac{17}{50} = 0.34$$

(18) What is the maximum curvature of the curve $y = \sqrt{3}\ln(x)$?

(A) $\frac{1}{9}$
[✓ (B) $\frac{2}{9}$]
(C) $\frac{1}{3}$
(D) $\frac{4}{9}$
(E) $\frac{5}{9}$
(F) $\frac{2}{3}$

$$\vec{r}(t) = \langle t, \sqrt{3}\ln t, 0\rangle$$

$$\vec{r}\,'(t) = \langle 1, \frac{\sqrt{3}}{t}, 0\rangle \Rightarrow \|\vec{r}\,'\| = \sqrt{1 + \frac{3}{t^2}}$$

$$\vec{r}\,''(t) = \langle 0, -\frac{\sqrt{3}}{t^2}, 0\rangle \Rightarrow \vec{r}\,'\times\vec{r}\,'' = \langle 0, 0, -\frac{\sqrt{3}}{t^2}\rangle$$

$$\Rightarrow \kappa = \frac{\sqrt{3}/t^2}{(1+3/t^2)^{3/2}} = \frac{t\sqrt{3}}{(t^2+3)^{3/2}}$$

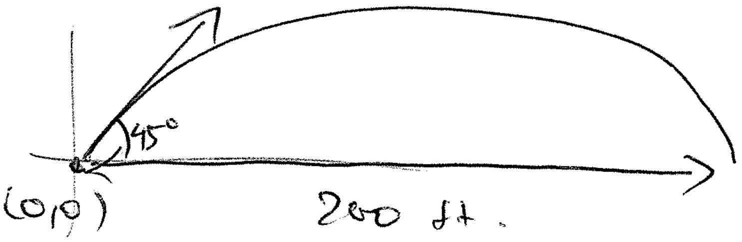
Set $0 = \kappa'(t) = \dfrac{(t^2+3)^{\cancel{3/2}}\sqrt{3} - t\sqrt{3}\,\cancel{\frac{3}{2}}\cancel{2}t\cdot\cancel{(t^2+3)}^{1/2}}{(t^2+3)^{\cancel{3}\,5/2}}$

$$= \sqrt{3}\,\frac{\cancel{t^2}+3-\overset{2}{\cancel{3}}t^2}{(t^2+3)^{5/2}} \Rightarrow 0 = 3 - 2t^2 \Rightarrow t = \sqrt{3/2}$$

and $\kappa(\sqrt{3/2}) = \dfrac{3/\sqrt{2}}{(3/2+3)^{3/2}} = \dfrac{3}{\sqrt{2}}\left(\dfrac{2}{9}\right)^{3/2} = \dfrac{3}{\cancel{\sqrt{2}}}\,\dfrac{2\cancel{\sqrt{2}}}{27} = \dfrac{2}{9}.$

(19) If a batted baseball leaves the bat at an angle of 45° with the horizontal, and is caught by a Cardinals outfielder 200 feet from home plate, what was the original velocity (in feet/second) of the ball? [Use $g = 32$ ft/sec^2]

(A) 55
(B) 60
(C) 65
(D) 70
(E) 75
[✓ (F) 80]



write $\vec{v_0} = \langle v_0 \frac{\sqrt{2}}{2}, v_0 \frac{\sqrt{2}}{2} \rangle$

$\vec{a} = \langle 0, -32 \rangle \Longrightarrow \vec{v} = \langle \frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}} - 32t \rangle$

$\Longrightarrow \vec{r} = \langle \frac{v_0 t}{\sqrt{2}}, \frac{v_0 t}{\sqrt{2}} - 16t^2 \rangle$

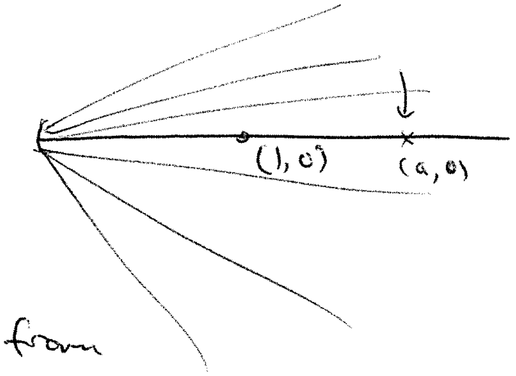
Set $\langle 200, 0 \rangle = \vec{r}(t_0) = \langle \frac{v_0 t_0}{\sqrt{2}}, \frac{v_0 t_0}{\sqrt{2}} - 16t_0^2 \rangle$

$$\Longrightarrow \begin{cases} \frac{v_0 \cancel{t_0}}{\sqrt{2}} = 16 t_0^{\cancel{2}} \to t_0 = \frac{v_0}{16\sqrt{2}} \\ \frac{v_0 t_0}{\sqrt{2}} = 200 \to \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{16\sqrt{2}} = 200 \to v_0^2 = 400 \cdot 16 \end{cases}$$

$\to v_0 = 20 \cdot 4 = 80.$

(20) Paris is located at the origin of the xy-plane. Rail lines emanate from Paris along all rays, and these are the only rail lines. (Yes, there are infinitely many.) Let $f(x, y)$ be the distance from (x, y) to $(1, 0)$ on the French railroad. Determine the set of points at which f is discontinuous.

[✓ (A) positive x-axis ($x > 0$, $y = 0$)]
(B) nonnegative x-axis ($x \geq 0$, $y = 0$)
(C) entire x-axis ($y = 0$)
(D) negative x-axis ($x < 0$, $y = 0$)
(E) the origin $(0, 0)$ only
(F) entire xy-plane



- limit of $f(x,y)$ as $(x,y) \to (a,0)$ from above or below is $a+1$
- limit as $(x,y) \to (a,0)$ along the (+)-x-axis is $|a-1|$. this is also $f(a,0)$.

So for $a > 0$, we do NOT have $\lim_{(x,y)\to(a,0)} f(x,y) = f(a,0)$

(in fact, the limit doesn't even exist!).