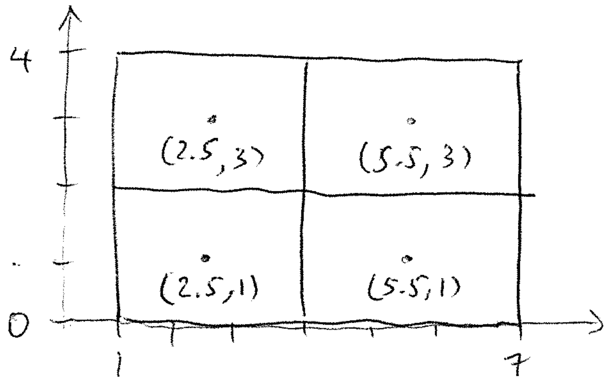
MATH 233 MIDTERM EXAM 3 SOLUTIONS

This exam consists of 14 multiple choice (machine-graded) problems; the first 6 are worth 6 points each and the last 8 are worth 8 points each (for a total of 100 points). No 3x5 cards or calculators are allowed. You will need a **pencil** to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit, and **shade in the corresponding boxes.** Also **print your name at the top of your card.**

Some possibly useful formulas: $\ln(x) - \ln(y) = \ln(x/y)$, $\sin^2\eta = 1 - \cos^2\eta = \frac{1}{2}(1 - \cos(2\eta))$.

(1) A pool situated in the rectangle $R = [1, 7] \times [0, 4]$ has depth $\frac{1}{8}xy^2$ as a function of (x, y). Use the midpoint rule with $m = n = 2$ to estimate the volume of the pool.
(A) 52
(B) 56
(C) 60
(D) 64
(E) 68
(F) 72



$$3 \cdot 2 \cdot \left(\frac{5}{16} + \frac{11}{16} + \frac{45}{16} + \frac{99}{16}\right) = \frac{3 \cdot \cancel{160}^{20}}{\cancel{8}} = 60.$$

Grade scale)

95-100	A+	56-59	C+
80-94	A	45-54	C
78-79	A-	40-44	C-
76-77	B+	30-39	D
64-72	B	below 30	F
60-63	B-		

(2) Compute the actual volume of the pool in problem (1).
(A) 52
(B) 56
(C) 60
(D) 64
(E) 68
(F) 72

$$\frac{1}{8}\int_1^7\int_0^4 xy^2\,dy\,dx = \frac{1}{8}\left[\frac{x^2}{2}\right]_1^7\left[\frac{y^3}{3}\right]_0^4$$
$$= \frac{1}{8}\left(\frac{49-1}{2}\right)\left(\frac{64}{3}\right) = 64$$

(3) Evaluate $\iint_D \frac{1}{4+x^2+y^2}\,dA$, where D is the disk $x^2 + y^2 \leq 4$.
(A) $\frac{\pi^2}{4}$
(B) $\pi\ln(2)$
(C) $\ln(2)$
(D) $\frac{\pi^2}{2}\ln(2)$
(E) $\pi\ln(4)$
(F) π^2

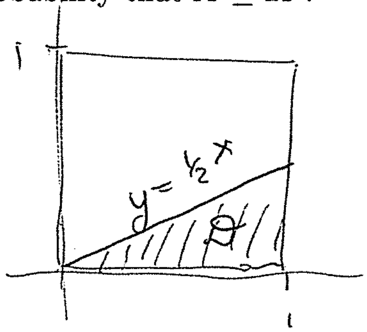
$$\int_0^{2\pi}\int_0^2 \frac{1}{4+r^2}\,r\,dr\,d\theta = 2\pi\int_0^2 \frac{r}{4+r^2}\,dr$$
$$= \cancel{2}\pi\left[\frac{1}{\cancel{2}}\ln(4+r^2)\right]_0^2$$
$$= \pi(\ln 8 - \ln 4)$$
$$= \pi\ln\frac{8}{4} = \pi\ln 2$$

(4) The joint density function for a pair of random variables X and Y is

$$f(x,y) = \begin{cases} 4xy, & \text{if } 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the probability that $X \ge 2Y$.

(A) $\frac{1}{2}$
(B) $\frac{1}{4}$
(C) $\frac{1}{8}$
(D) $\frac{1}{16}$
(E) $\frac{1}{32}$
(F) $\frac{1}{64}$



$$P = \iint_D 4xy \, dA = \int_0^1 \int_0^{\frac{x}{2}} 4xy \, dy \, dx$$
$$= \int_0^1 \left[2xy^2\right]_{y=0}^{y=x/2} dx = \frac{1}{2}\int_0^1 x^3 \, dx$$
$$= \frac{1}{8}.$$

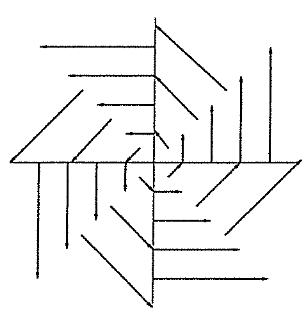
(5) If we want to change variables in a double integral from (x,y) to (u,v), where $x = 2u+3v$ and $y = \frac{1}{2}(v^2 - u^2)$, we must replace $dx\,dy$ by what function times $du\,dv$?

(A) $2u+3v$
(B) $3u-2v$
(C) $6uv$
(D) $2u-3v$
(E) $3u+2v$
(F) $3uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} 2 & 3 \\ -u & v \end{pmatrix}$$
$$= 2v + 3u$$

(6) Which vector field is sketched in the picture?

(A) $-y\hat{\imath} + x\hat{\jmath}$
(B) $x\hat{\imath}$
(C) $-y\hat{\imath}$
(D) $-x\hat{\jmath}$
(E) $x\hat{\imath} + y\hat{\jmath}$
(F) $y\hat{\jmath}$



(7) Compute $\iint_R \frac{xy}{\sqrt{1+x^2y}}dA$, where $R = [0,1]\times[0,3]$.

(A) $\frac{1}{3}$
(B) $\frac{2}{3}$
(C) 1
(D) $\frac{4}{3}$
(E) $\frac{5}{3}$
(F) 2

$$\int_0^3 \int_0^1 xy\,(1+x^2y)^{-1/2}\,dx\,dy$$

$$= \int_0^3 \left[(1+x^2y)^{1/2}\right]_{x=0}^{1} dy$$

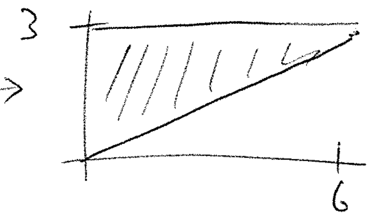
$$= \int_0^3 \left\{(1+y)^{1/2} - 1\right\} dy$$

$$= \left[\frac{2}{3}(1+y)^{3/2} - y\right]_0^3$$

$$= \left(\frac{2}{3}\cdot 8 - 3\right) - \left(\frac{2}{3} - 0\right)$$

$$= \frac{14}{3} - \frac{9}{3} = \frac{5}{3}$$

(8) Compute the iterated integral $\int_0^6 \int_{\frac{x}{2}}^3 x\,e^{y^3}\,dy\,dx$.



(A) $\frac{2}{3}(e^{27}-1)$
(B) $\frac{1}{3}(e^{27}-1)$
(C) $\frac{1}{3}e^{27}$
(D) $\frac{2}{3}$
(E) $\frac{2}{3}(e^9-1)$
(F) $\frac{1}{3}e^9$

$$= \int_0^3 \int_0^{2y} x\,e^{y^3}\,dx\,dy$$

$$= \int_0^3 e^{y^3}\left[\frac{x^2}{2}\right]_{x=0}^{2y} dy$$

$$= 2\int_0^3 y^2 e^{y^3}\,dy = \frac{2}{3}\,e^{y^3}\Big|_0^3$$

$$= \frac{2}{3}\left(e^{27}-1\right)$$

(9) Find the area of one leaf of the 4-leaved rose $r = 3\sin 2\theta$. $\left(\theta \in [0, \pi/2] \right)$

(A) π
(B) $\frac{\pi}{2}$
(C) $\frac{3\pi}{8}$
(D) $\frac{9\pi}{8}$
(E) $\frac{9-\pi}{8}$
(F) $\frac{\pi-1}{8}$

$$\int_0^{\pi/2}\int_0^{3\sin 2\theta} \underbrace{r\,dr\,d\theta}_{dA} = \frac{9}{2}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{9}{4}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

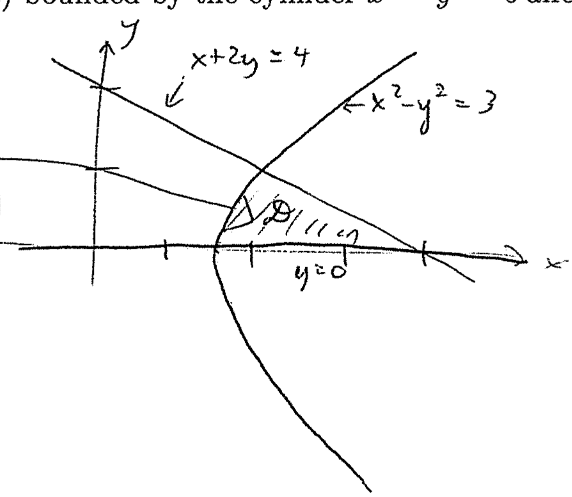
$$= \frac{9}{4}\left[\theta - \cancel{\tfrac{1}{4}\sin 4\theta}\right]_0^{\pi/2}$$

$$= \frac{9}{4}\cdot\frac{\pi}{2} = \frac{9\pi}{8}.$$

(10) Find the volume of the solid (in the first octant) bounded by the cylinder $x^2 - y^2 = 3$ and the planes $z = 0$, $3y = z$, $y = 0$, and $x + 2y = 4$.

(A) $5 - 2\sqrt{3}$
(B) $4 - \sqrt{3}$
(C) $3\sqrt{3} + 4$
(D) $4\sqrt{3} - 5$
(E) $4 + \sqrt{3}$
(F) $3\sqrt{3} - 4$

(Solid given by $z \in [0, 3y]$ over $\mathcal{D}$)



$$V = \int_0^1 \int_{\sqrt{3+y^2}}^{4-2y} 3y \, dx\, dy$$

$$= 3\int_0^1 y\left(4 - 2y - \sqrt{3+y^2}\right) dy$$

$$= \left[6y^2 - 2y^3 - (3+y^2)^{3/2}\right]_0^1$$

$$= (6 - 2 - 8) - (0 - 0 - 3\sqrt{3})$$

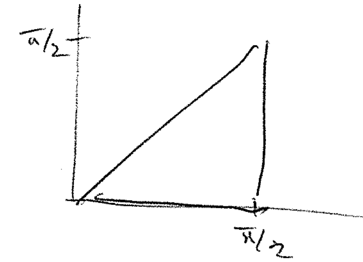
$$= 3\sqrt{3} - 4$$

(11) Find the average value of $\cos(y)$ on the triangular region with vertices $(0,0)$, $(\frac{\pi}{2},0)$ and $(\frac{\pi}{2},\frac{\pi}{2})$.

(A) $\frac{\pi}{4}$
(B) $\frac{8}{\pi^2}$
(C) $\frac{4}{\pi}$
(D) $\frac{1}{2}$
(E) $\frac{\sqrt{3}}{2}$
(F) $\frac{\pi^2}{8}$

$$\int_0^{\pi/2}\int_0^x \cos y \, dy \, dx = \int_0^{\pi/2} \sin x \, dx$$

$$= \left[-\cos x\right]_0^{\pi/2} = 1$$



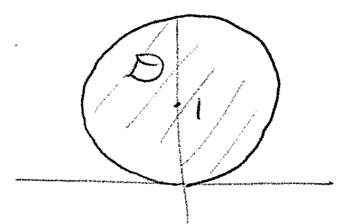
$$\text{area} = \frac{1}{2} b h = \frac{1}{2}\frac{\pi}{2}\frac{\pi}{2} = \frac{\pi^2}{8}$$

So avg. value is $\frac{\iint}{\text{area}} = \frac{1}{\pi^2/8} = \frac{8}{\pi^2}$

(12) Find the center of mass of the circular lamina bounded by $r = 2\sin\theta$ $(0 \le \theta \le \pi)$ with density function $\rho(r,\theta) = r$.

(A) $(0,1)$
(B) $(0,\frac{6}{5})$
(C) $(0,\frac{4}{3})$
(D) $(0,\frac{7}{5})$
(E) $(0,\frac{5}{4})$
(F) $(0,\frac{8}{5})$

$$M_x = \iint_D y\,\rho(r,\theta)\,dA$$



$$= \int_0^{\pi}\int_0^{2\sin\theta} r\sin\theta \; r \, dr \, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^4}{4}\right]_0^{2\sin\theta} d\theta$$

$$= 4\int_0^{\pi} \sin^5\theta \, d\theta = 4\int_0^{\pi} \sin\theta\,(1-\cos^2\theta)^2 \, d\theta$$

$$= 4\int_0^{\pi} \sin\theta\,(1 - 2\cos^2\theta + \cos^4\theta)\, d\theta$$

$$= 4\left[-\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta\right]_0^{\pi}$$

$$= 4\cdot 2\cdot\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 8\,\frac{15-10+3}{15} = \frac{64}{15}.$$

$$m\ (= \text{mass}) = \iint_D \rho(r,\theta)\,dA = \int_0^{\pi}\int_0^{2\sin\theta} r^2 \, dr \, d\theta = \dots = \frac{32}{9}$$

$$\bar{y} = M_x/m = \frac{64/15}{32/9} = 6/5.$$

(13) Find the maximum of $f(x,y,z) = 4x - 2y + 3z$, subject to the constraint $2x^2 + y^2 + 3z^2 = 15$.

(A) 6
(B) 9
(C) 12
(D) 15
(E) 18
(F) 21

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\langle 4,-2,3\rangle = \lambda \langle 4x, 2y, 6z\rangle$$

$$4 = 4\lambda x, \quad -2 = 2\lambda y, \quad 3 = 6\lambda z$$

$\Rightarrow$ none of $x, y, z, \lambda = 0$

$$\Rightarrow x = \frac{1}{\lambda}, \quad y = -\frac{1}{\lambda}, \quad z = \frac{1}{2\lambda}.$$

$$\Rightarrow 15 = \frac{2}{\lambda^2} + \frac{1}{\lambda^2} + \frac{3}{4\lambda^2} = \frac{8+4+3}{4\lambda^2} = \frac{15}{4\lambda^2}$$

$$\Rightarrow \lambda = \pm \frac{1}{2}.$$

So $x = \pm 2, \ y = \mp 2, \ z = \pm 1$.

$f(2,-2,1) = 8+4+3 = 15$.

$f(-2,2,-1) = -15$.

(14) Find the minimum distance between the xy-plane and the curve cut out by $4x - 3y + 8z = 5$ and $z^2 = x^2 + y^2$.

(A) $\frac{6}{13}$
(B) $\frac{5}{13}$
(C) $\frac{4}{13}$
(D) $\frac{3}{13}$
(E) $\frac{2}{13}$
(F) $\frac{1}{13}$

function describing distance $= z =: f(x,y,z)$

$$\vec{\nabla} f = \lambda \vec{\nabla} g + \mu \vec{\nabla} h$$

$$\langle 0,0,1\rangle = \lambda\langle 4,-3,8\rangle + \mu\langle x, y, -z\rangle$$

$$0 = 4\lambda + \mu x, \quad 0 = -3\lambda + \mu y, \quad 1 = 8\lambda - \mu z$$

$$x = -\frac{4\lambda}{\mu}, \quad y = \frac{3\lambda}{\mu} = -\frac{3}{4}x$$

$$z^2 = x^2 + y^2 = x^2 + \frac{9}{16}x^2 = \frac{25}{16}x^2 \Rightarrow z = \pm\frac{5}{4}x$$

So $5 = 4x + 3\cdot\frac{3}{4}x \pm 8\cdot\frac{5}{4}x \Rightarrow 20 = (16+9\pm 40)x$

$$\Rightarrow x = \frac{4}{13} \text{ or } -\frac{4}{3}.$$

$$\Rightarrow z = \frac{5}{13} \text{ or } \frac{5}{3}.$$