MATH 233 FINAL EXAM SOLUTIONS

This exam consists of 25 multiple choice (machine-graded) problems, worth 4 points each. No 3x5 cards or calculators are allowed. You will need a **pencil** to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit, and **shade in the corresponding boxes.** Also **print your name at the top of your card.** Good luck!

Some possibly useful formulas: $\ln(x) - \ln(y) = \ln(x/y)$, $\sin^2\eta = 1 - \cos^2\eta = \frac{1}{2}(1 - \cos(2\eta))$.

(1) Find the area of the triangle in space with coordinates $(1, 2, -3)$, $(5, -1, 1)$, and $(1, 3, -2)$.

(A) 4
(B) $\frac{9}{2}$
(C) 5
(D) $\frac{11}{2}$
(E) 6
(F) $\frac{13}{2}$

$$\frac{1}{2}\|\langle 4,-3,4\rangle \times \langle 0,1,1\rangle\| = \frac{1}{2}\|\langle -7,-4,4\rangle\|$$
$$= \frac{1}{2}\sqrt{49+16+16} = \frac{1}{2}\sqrt{81} = \frac{9}{2}$$

GRADE SCALE

A+	96-100	B+	72	C+	52	D	28-32
A	80-92	B	60-68	C	40-48	F	0-24
A−	76	B−	56	C−	36		

(2) Let P be the plane which contains the triangle of problem (1), and Q the plane with equation $x - 2y + 2z = 6$. What is the cosine of the angle between P and Q?

(A) 0
(B) $\frac{1}{6}$
(C) $\frac{1}{4}$
(D) $\frac{1}{3}$
(E) $\frac{1}{2}$
(F) 1

normal vectors

$$\cos\theta = \frac{\langle -7,-4,4\rangle \cdot \langle 1,-2,2\rangle}{\|\langle -7,-4,4\rangle\| \, \|\langle 1,-2,2\rangle\|} = \frac{9}{9\cdot 3} = \frac{1}{3}$$

(3) Evaluate $f_{xy}(1, 0)$ for $f(x, y) = \sin(x^2y)$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

$$f_x = 2xy\cos(x^2y)$$
$$(f_x)_y = 2x\cos(x^2y) - 2x^3y\sin(x^2y)$$
$$f_{xy}(1,0) = 2\cdot 1\cdot\cos 0 - \cancel{2\cdot 1\cdot 0\cdot \sin 0}^{\,0}$$
$$= 2$$

(4) Determine the steepest (i.e. maximum) slope of $z = f(x,y) = x^3 + xy + 4y$ at $(x,y) = (2,-4)$.

(A) 5
(B) 6
(C) 7
(D) 8
(E) 9
(F) 10

$$\vec{\nabla} f = \langle 3x^2 + y,\ x + 4 \rangle$$

$$\| \vec{\nabla} f(2,-4) \| = \| \langle 8, 6 \rangle \| = \sqrt{64+36} = 10$$

(5) Evaluate $\iint_D \frac{1}{x^2+y^2} dA$ where D is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

(A) $\pi \ln(2)$
(B) $\frac{\pi}{3}$
(C) π
(D) $\pi \ln(\frac{3}{2})$
(E) $2\pi \ln(\frac{3}{2})$
(F) $\frac{\pi}{2}$

$$\int_0^{2\pi} \int_2^3 \frac{1}{r^2}\, r\, dr\, d\theta = 2\pi \int_2^3 \frac{1}{r}\, dr$$

$$= 2\pi(\ln(3) - \ln(2))$$

$$= 2\pi \ln(3/2)$$

(6) Calculate the iterated integral $\int_0^1 \int_x^{\sqrt{x}} 4xy\, dy\, dx$.

(A) 0
(B) $\frac{1}{6}$
(C) $\frac{1}{4}$
(D) $\frac{1}{3}$
(E) $\frac{1}{2}$
(F) 1

$$= \int_0^1 \left[2xy^2 \right]_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (2x^2 - 2x^3)\, dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{2} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

(7) Suppose we are integrating in x and y and we want to integrate in u and v, where $x = \frac{u^2}{v}$ and $y = \frac{v^4}{u}$. We must replace $dx\,dy$ by what function times $du\,dv$?

(A) $7v^2 + 5u^2$
(B) $9u^2$
(C) $7v^2$
(D) $4v^2 + 7u^2$
(E) $5u^2$
(F) $9v^2$

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 2u/v & -u^2/v^2 \\ -v^4/u^2 & 4v^3/u \end{vmatrix} = \frac{2u}{v}\frac{4v^3}{u} - \frac{u^2}{v^2}\frac{v^4}{u^2}$$

$$= 8v^2 - v^2$$

$$= 7v^2$$

(8) For which of the vector fields

$$\text{(i) } y\mathbf{i} + x\mathbf{j}, \qquad \text{(ii) } -y\mathbf{i} + x\mathbf{j}, \qquad \text{(iii) } \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

is the line integral *zero* along any closed path encircling the origin?

(A) (i) only
(B) (ii) only
(C) (iii) only
(D) (i) and (ii)
(E) (ii) and (iii)
(F) (i) and (iii)

check curl $= Q_x - P_y$:

0 for (i) & (iii) ;

but (iii) has a singularity (and is our example from class). So only (i) is conservative.

(9) Let f be a function (scalar field) and $\vec{F}$ a vector field, both on $\mathbb{R}^3$. How many of

$$\vec{\nabla} f, \quad \mathrm{curl}(\vec{F}), \quad \mathrm{div}(\vec{F}), \quad \mathrm{div}(\mathrm{div}(\vec{F})), \quad \mathrm{curl}(\mathrm{curl}(\vec{F})), \quad \mathrm{curl}(\mathrm{div}(\vec{F}))$$

are: scalar fields; vector fields; meaningless?

(A) 3;2;1
(B) 2;1;3
(C) 1;3;2
(D) 2;3;1
(E) 3;1;2
(F) 1;2;3

(10) How many of the vector fields

$$3y^2z\mathbf{i} + (y - x^3)\mathbf{k}, \qquad 3y^2z\mathbf{i} + xy\mathbf{j} - x^3\mathbf{k}, \qquad 3y^2z\mathbf{i} + x^3\mathbf{j} + yz\mathbf{k},$$

$$(3y^2z - xy)\mathbf{i} + x^3\mathbf{j} + yz\mathbf{k}, \qquad -xy\mathbf{i} + \tfrac{1}{2}y^2\mathbf{j} + e^{x^2y^2}\mathbf{k}$$

on $\mathbb{R}^3$ are the curl of another vector field?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

check $\underline{\underline{\text{div}}}$: $0, \quad x, \quad y$

$0, \quad 0$

need $= 0$.

(11) Determine the value of a (if any) that makes $\vec{F}(x,y,z) = y^2\mathbf{i} + (axy + e^{3z^2})\mathbf{j} + 3(ayz)e^{3z^2}\mathbf{k}$ a conservative field on $\mathbb{R}^3$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

need: $P_y = Q_x \rightsquigarrow 2y = ay$

$P_z = R_x \rightsquigarrow 0 = 0$

$Q_z = R_y \rightsquigarrow 6ze^{3z^2} = 3aze^{3z^2}$

so $a = 2$

(12) Describe the flow lines of $\vec{F}(x,y) = x\mathbf{i} - y\mathbf{j}$.

(A) parabolas
(B) hyperbolas
(C) circles
(D) ellipses
(E) lines
(F) none of these

$x'(t) = x(t) \rightsquigarrow x(t) = Ce^{t}$

$y'(t) = -y(t) \rightsquigarrow y(t) = Ke^{-t} = \dfrac{C_0}{x(t)}$

$y = \dfrac{C_0}{x}$.

(13) Evaluate the line integral $\int_C xe^{xy}ds$, where C is the line segment from $(0,0)$ to $(3,4)$. $(0 \le t \le 1)$

(A) $\frac{1}{8}(e^3-1)$
(B) $\frac{1}{8}(e^4-1)$
(C) $\frac{1}{8}(e^{12}-1)$
(D) $\frac{5}{8}(e^3-1)$
(E) $\frac{5}{8}(e^4-1)$
(F) $\frac{5}{8}(e^{12}-1)$

$\vec{r}(t) = \langle 3t, 4t \rangle$

$\|\vec{r}'(t)\| = 5$

$$= \int_0^1 3te^{12t^2} \cdot 5\,dt$$
$$= 15\int_0^1 te^{12t^2}\,dt$$
$$= \frac{15}{24}\left[e^{12t^2}\right]_0^1$$
$$= \frac{5}{8}(e^{12}-1).$$

(14) Find the work done by the force field $\vec{F}(x,y) = x^3\mathrm{i}+2x^2y\mathrm{j}$ in moving an object along the quarter-circle of radius 1 (centered at $(0,0)$) from $(1,0)$ to $(0,1)$.

(A) 0
(B) $\frac{1}{6}$
(C) $\frac{1}{4}$
(D) $\frac{1}{3}$
(E) $\frac{1}{2}$
(F) 1

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \cos^3 t\,(-\sin t\,dt) + 2\cos^2 t \sin t\,(\cos t\,dt)$$
$$= \int_0^{\pi/2} \cos^3 t \sin t\,dt$$
$$= -\frac{1}{4}\left[\cos^4 t\right]_0^{\pi/2}$$
$$= 1/4$$

(15) Let C be an arbitrary (possibly curved) path from $(1,1)$ to $(2,3)$. Compute the line integral of $\vec{F}(x,y) = (4x^3-2xy)\mathrm{i}+(2-x^2)\mathrm{j}$ along C.

(A) 0
(B) 2
(C) 4
(D) 6
(E) 8
(F) 10

$\vec{F} = \vec{\nabla} f$, where

$f = x^4 - x^2y + 2y$

$$\int_C \vec{\nabla} f\cdot d\vec{r} = f(2,3) - f(1,1)$$
$$= 2^4 - 2^2\cdot 3 + 2\cdot 3 - (1-1+2)$$
$$= 8.$$

(16) What is the integral of the vector field $\vec{F}(x,y) = (yx^2e^{x^3}+5y)\mathbf{i}+(\frac{1}{3}e^{x^3}+8x)\mathbf{j}$ over the (counterclockwise) circle C of radius 2 (centered about $(0,0)$)? [Hint: you won't be able to compute this line integral directly. Use a theorem.]

P Q

(A) 2π
(B) 4π
(C) 6π
(D) 8π
(E) 10π
(F) 12π

Green's thm.

$Q_x = x^2e^{x^3}+8$

$P_y = x^2e^{x^3}+5$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{(Q_x - P_y)}_{=3}\, dA$$

$$= 3\underbrace{A(S)}_{\pi(2)^2}$$

$$= 12\pi$$

(17) Find the area of the surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 1$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

$$\vec{r}_u \times \vec{r}_v = \langle 2u, v, 0\rangle \times \langle 0, u, v\rangle$$

$$= \langle v^2, -2uv, 2u^2\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^4+4u^2v^2+4u^4} = \sqrt{(v^2+2u^2)^2}$$

$$= v^2+2u^2$$

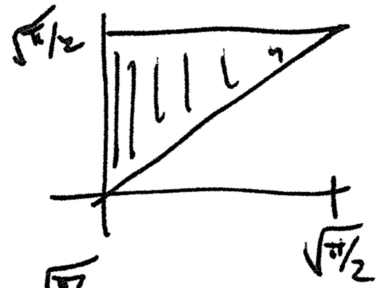
$$A = \int_0^1\int_0^1 (v^2+2u^2)\,du\,dv$$

$$= \int_0^1 2u^2\,du + \int_0^1 v^2\,dv$$

$$= \frac{2}{3}u^3\Big|_0^1 + \frac{v^3}{3}\Big|_0^1 = \frac{2}{3}+\frac{1}{3} = 1$$

(18) Calculate the iterated integral $\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \cos(y^2)\,dy\,dx$. [Hint: reverse the order.]

(A) 0
(B) $\frac{1}{6}$
(C) $\frac{1}{4}$
(D) $\frac{1}{3}$
(E) $\frac{1}{2}$
(F) 1



$$= \int_0^{\sqrt{\pi/2}} \int_0^y \cos(y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{\pi/2}} y\cos(y^2)\,dy$$

$$= \frac{1}{2}\left[\sin(y^2)\right]_0^{\sqrt{\pi/2}}$$

$$= \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$$

(19) Describe the curve $r = \tan\theta\sec\theta$.

(A) parabola
(B) hyperbola
(C) circle
(D) ellipse
(E) line
(F) none of these

$$r = \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} = \frac{r\sin\theta}{r\cos\theta} \cdot \frac{r}{r\cos\theta}$$

$$\cancel{r} = \frac{y}{x} \cdot \frac{\cancel{r}}{x}$$

$$x^2 = y$$

(20) Determine the arclength of the curve traced out by $\vec{r}(t) = \langle \frac{1}{3}t^3, t^2, 2t \rangle$ for $0 \le t \le 3$.

(A) 9
(B) 12
(C) 15
(D) 18
(E) 21
(F) 24

$\|\cdot\|$ $\vec{r}'(t) = \langle t^2, 2t, 2 \rangle$

$$\int_0^3 \sqrt{t^4 + 4t^2 + 4}\, dt$$

$$= \int_0^3 (t^2 + 2)\, dt$$

$$= \left[t^3/3 + 2t \right]_0^3$$

$$= \frac{3^3}{3} + 2 \cdot 3 = 9 + 6 = 15$$

(21) For $\vec{r}(t)$ as in problem (20), compute the radius of the osculating circle at $t = 0$.

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5
(F) 6

$\vec{r}'(0) = \langle 0, 0, 2 \rangle$

$\vec{r}'' = \langle 2t, 2, 0 \rangle$

$\vec{r}''(0) = \langle 0, 2, 0 \rangle$

$\vec{r}'(0) \times \vec{r}''(0) = \langle -4, 0, 0 \rangle$

$$R(0) = \frac{1}{\kappa(0)} = \frac{\|\vec{r}'(0)\|^3}{\|\vec{r}'(0) \times \vec{r}''(0)\|}$$

$$= \frac{2^3}{4} = 2$$

(22) A triangle has vertices A, B, and C. The length of the side AB is 10 inches, and is increasing at a rate of 3 inches/second. The length of the side AC is 8 inches, and is *decreasing* at a rate of 4 inches/second. The angle θ between AB and AC is $\frac{\pi}{6}$ and increasing at $\frac{\sqrt{3}}{10}$ radians/second. How fast (in in^2/sec) is the area changing?
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

$$\text{Area} = \frac{1}{2}\|\vec{AB} \times \vec{AC}\| = \frac{1}{2}\|\vec{AB}\|\|\vec{AC}\| \sin\theta = \frac{1}{2} xy \sin\theta$$

$x = 10, \quad y = 8, \quad \theta = \pi/6$

$x' = 3, \quad y' = -4, \quad \theta' = \sqrt{3}/10$

$$A' = A_x x' + A_y y' + A_\theta \theta'$$
$$= \frac{1}{2}(y\sin\theta)\cdot x' + \frac{1}{2}(x\sin\theta)y' + \frac{1}{2}(xy\cos\theta)\theta'$$
$$= \frac{1}{2}\left(8\cdot\frac{1}{2}\right)3 + \frac{1}{2}\left(10\cdot\frac{1}{2}\right)(-4) + \frac{1}{2}\left(10\cdot 8\cdot\frac{\sqrt{3}}{2}\right)\frac{\sqrt{3}}{10}$$
$$= 6 - 10 + 6$$
$$= 2$$

(23) How many of each kind of critical point does $f(x, y) = xy^2 - 6x^2 - 3y^2$ have: local maximum; local minimum; saddle point?
(A) 1;0;1
(B) 0;1;1
(C) 2;0;1
(D) 1;1;1
(E) 1;0;2
(F) 0;1;2

$0 = f_x = y^2 - 12x$

$0 = f_y = 2xy - 6y = 2y(x-3)$

$\Downarrow$

- $y = 0 \Rightarrow x = 0$

OR

- $x = 3 \Rightarrow y = \pm 6$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -12 & 2y \\ 2y & 2x-6 \end{vmatrix} = -4y^2 - 24x + 72$$

$D(0,0) = 72 > 0, \quad f_{xx}(0,0) = -12 < 0 \quad \leadsto$ local max

$D(3, \pm 6) = -72 + 72 - 4\cdot 6^2 < 0 \quad \leadsto$ 2 saddle pts.

(24) Find the maximum of $f(x,y) = xy$, subject to the constraint $g(x,y) = 4x^2 + 9y^2 - 36 = 0$.
(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$\langle y, x \rangle = \lambda \langle 8x, 18y \rangle$$

$$y = 8\lambda x, \quad x = 18\lambda y = 18\lambda \cdot 8\lambda x = 9 \cdot 16 \lambda^2 x$$

$$\Rightarrow x = 0 \text{ or } \lambda = \pm \tfrac{1}{12}.$$

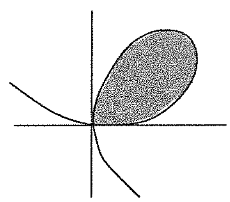
impossible, since it gives $y = 0$ and $(0,0)$ doesn't satisfy constraint.

plug $y = \pm \frac{8}{12} x = \pm \frac{2}{3} x$ into constraint:

$$36 = 4x^2 + 9 \cdot \tfrac{4}{9} x^2 = 8x^2 \Rightarrow x = \pm \tfrac{3\sqrt{2}}{2}, \quad y = \pm\sqrt{2}$$

$$\Rightarrow xy = \tfrac{3\sqrt{2}}{2} \cdot \sqrt{2} = 3.$$

(25) The folium of Descartes is the curve $x^3 + y^3 = 3xy$ in the plane.



The portion in the first quadrant may be parametrized by $\vec{r}(t) = \underbrace{\frac{3t}{1+t^3}}_{x(t)}\mathrm{i} + \underbrace{\frac{3t^2}{1+t^3}}_{y(t)}\mathrm{j}$, as t runs from 0 to $+\infty$. Use Green's theorem to find the area it encloses.
(A) $\frac{1}{2}$
(B) 1
(C) $\frac{3}{2}$
(D) 2
(E) $\frac{5}{2}$
(F) 3

$$y'(t) = \frac{3t(2-t^3)}{(1+t^3)^2}$$

$$x'(t) = \frac{3(1-2t^3)}{(1+t^3)^2}$$

$$A(R) = \frac{1}{2} \oint_{\partial R} -y\,dx + x\,dy$$

$$= \frac{1}{2} \int_0^\infty \left\{ -\frac{3t^2}{(1+t^3)} \cdot \frac{3-6t^3}{(1+t^3)^2} + \frac{3t}{(1+t^3)} \cdot \frac{6t-3t^4}{(1+t^3)^2} \right\} dt$$

$$= \frac{3}{2} \int_0^\infty \frac{3t^2 \cancel{(t^3+1)}}{(1+t^3)^{\cancel{3}\,2}}\,dt$$

$$= \frac{3}{2} \left[\frac{-1}{(1+t^3)} \right]_0^\infty = \frac{3}{2}.$$