MATH 233 LECTURE 10 (§13.3): CURVATURE AND OSCULATING CIRCLES

Review of notation: $\vec{r}(t)$ is a vector valued function (position), $\vec{r}'(t)$ the velocity (tangent vector), $\|\vec{r}'(t)\| = \frac{ds}{dt}$ the speed, and $\vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ the direction (unit tangent vector). The curvature is $\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$, which becomes simply $\kappa(t) = \|\vec{r}''(t)\|$ if $\vec{r}(t)$ has speed constantly equal to 1 (parametrization by arclength).

More curvature formulas.

- Another general formula is $\kappa(t) = \|\vec{r}'(t) \times \vec{r}''(t)\| \,/\, \|\vec{r}'(t)\|^3$, which does away with differentiating $\vec{T}$ (usually no fun). It is based on the idea that $\vec{T}(t)$ (being of unit length) lies on the surface of a sphere, hence has $\vec{T}(t) \perp \vec{T}'(t)$. (Details in Stewart, and in lecture.)
- A convenient formula for the special case where $\vec{r}(t) = \langle t, f(t), 0\rangle$ traces out the graph of a function $y = f(x)$ (in the xy-plane) is $\kappa(t) = |f''(t)|/\{1+(f'(t))^2\}^{3/2}$.

Osculating planes and circles.

- The curve C traced out by $\vec{r}(t)$ has (at each point) one tangent direction and *two* normal directions. So we need two normal vectors. The first is the "direction of curvature (or acceleration)": the *principal unit normal vector* $\vec{N}(t) := \vec{T}'(t)/\left\|\vec{T}'(t)\right\|$. The second is the *secondary* or *binormal unit normal vector* $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.
- The *normal plane* to C (at $\vec{r}(t)$) contains $\vec{B}(t)$ and $\vec{N}(t)$, and has normal vector $\vec{T}(t)$. The *osculating plane* (at $\vec{r}(t)$) contains $\vec{T}(t)$ and $\vec{N}(t)$ (as well as the osculating circle), and has normal vector $\vec{B}(t)$. Note that having the normal vectors gives you the information you need to write down equations for these planes.

- Just as we wrote parametric equations for the tangent lines to C, we can write equations for the osculating circles. The parametrization for an arbitrary circle is $\vec{\mathfrak{r}}(u) = \vec{C} + (\cos u)\vec{A} + (\sin u)\vec{B}$. To find the center $\vec{C}$, use the fact that the osculating circle at $\vec{r}(t)$ has radius $\frac{1}{\kappa(t)}$ and $\vec{N}(t)$ points toward the center. This gives $\vec{C} = \vec{r}(t) + \frac{1}{\kappa(t)}\vec{N}(t)$, and taking $\vec{A} = \frac{1}{\kappa(t)}\vec{T}(t)$ and $\vec{B} = \frac{1}{\kappa(t)}\vec{N}(t)$ finishes the job.