MATH 233 LECTURE 15 (§14.4): PARTIAL DERIVATIVES AND TANGENT PLANES

- Given a point P on a surface S in $\mathbb{R}^3$, there are lots of tangent lines through that point. The *tangent plane* to S at P contains them all.
- If the surface is $z = f(x, y)$, and the point is (x_0, y_0, z_0) (with $z_0 = f(x_0, y_0)$) the formula for the tangent plane is

$$z - z_0 = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0).$$

  It contains the tangent lines in the x and y directions that we considered in connection with partial derivatives.
- We can consider the tangent plane as the graph (i.e. $z = L(x, y)$) of the *linear approximation*

$$L(x, y) := f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)$$

  to $f(x, y)$.
- When does this really give a good approximation to f near (x_0, y_0)? That is, when do we have

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - L(x, y)}{d\left((x, y), (x_0, y_0)\right)} = 0?$$

  In this case, we say f is *differentiable* at (x_0, y_0).
- If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0). It also turns out to be true that *if f_x and f_y are continuous at (x_0, y_0), then f is differentiable there*. (This is the main fact you will need to keep in mind.) It certainly isn't true that all continuous functions are differentiable: continuous functions can have corners and spikes where the partials don't even exist; and there is

the more subtle situation where the partials exist but aren't continuous (for example, consider the function $f(x,y) = \frac{x^3}{x^2+y^2}$, which is continuous at $(0,0)$, and whose f_x and f_y exist there, yet which is not differentiable there). Anyway, for most functions you will meet this isn't an issue.

- Differentials: if $z = f(x,y)$, consider the first formula for the tangent plane above, and replace $z - z_0$, $x - x_0$, $y - y_0$ by tiny changes dz, dx, dy. Then we get

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

  This dz is really the change in L, i.e. approximate change in f, brought about by the changes in x and y; the book writes Δz for the actual change in f.

- Error propagation: this is just a reinterpretation of the last formula, viewing the changes dx and dy as the maximum possible error in two measured quantities x_0 and y_0; then dz is the (approximate) maximum possible error in the *computed* value of $z_0 = f(x_0, y_0)$.