MATH 233 LECTURE 16 (§14.5): THE MULTIVARIABLE CHAIN RULE

- For functions of 1 variable, recall how this goes: let $z = f(x)$ be a function and $x = g(t)$; so $z = f(g(t))$. Then

$$\frac{d}{dt} f(g(t)) = \frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} = f'(g(t)) \cdot g'(t).$$

- For functions of 2 variables, first suppose $z = f(x, y)$ and $x = g(t)$, $y = h(t)$; so $z = f(g(t), h(t))$. Then

$$\frac{d}{dt} f(g(t), h(t)) = \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = f_x(g(t), h(t)) \cdot g'(t) + f_y(g(t), h(t)) \cdot h'(t).$$

  The intuition here is that *both* the change in x and the change in y contribute to the change in z. I will give some explanation of this in class.

- More complicated variants: if $z = f(x, y)$ and $x = g(u, v)$, $y = h(u, v)$, then you get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

  which I won't write out in terms of the functions. Of course, there are versions involving more variables and higher partials.

- It is helpful to draw a sort of flow-chart depicting the dependencies of various variables the first few times you work problems involving these chain rules. Main point is that you have some independent variables (e.g. u and v above), some intermediate variables (e.g. x and y above), and the dependent variable(s) (e.g. z in the above).