MATH 233 LECTURE 18 (§14.6): DIRECTIONAL DERIVATIVES (CONT.)

- Recall that given $f(x, y)$ differentiable, the gradient is given by $\vec{\nabla} f = \langle f_x, f_y \rangle$, and points normal to the level curves of f. Given a unit vector $\hat{u}$, the directional derivative is $D_{\hat{u}} f = (\vec{\nabla} f) \cdot \hat{u}$; it gives the slope of f in the direction $\hat{u}$. This is largest in the direction of $\vec{\nabla} f$.
- Let L be the tangent line to the level curve C through $P(x_0, y_0)$. One way to find the equation of L is to parametrize C and use the formula (for the parametric equation) from Lecture 8. But it may be hard to find a parametrization. Since $\vec{n} := (\vec{\nabla} f)(x_0, y_0) = \langle a, b \rangle$ is normal to C (hence L), any point $Q(x, y)$ on L satisfies $0 = \overrightarrow{PQ} \cdot (\vec{\nabla} f)(x_0, y_0) = a(x - x_0) + b(y - y_0)$.
- For a function $F(x, y, z)$ of 3 variables, the same reasoning yields an equation for the tangent plane to the level surface through (x_0, y_0, z_0) (with $\vec{n} = (\vec{\nabla} F)(x_0, y_0, z_0)$).
- There are more uses of this observation about the gradient. Suppose you have two functions $f(x, y)$, $g(x, y)$ with level curves C_f, C_g through (x_0, y_0). When are these curves tangent? i.e. when do they have the same tangent line? Well, when their normal vectors are parallel! That is, when $(\vec{\nabla} f)(x_0, y_0)$ is a scalar multiple of $(\vec{\nabla} g)(x_0, y_0)$.
- Again, you can do the same thing with level surfaces S_F, S_G of functions $F(x, y, z)$, $G(x, y, z)$ in $\mathbb{R}^3$: they are tangent (share the same tangent plane) at a point if the gradients are multiples of each other there.
- Likewise, you can say that two curves or surfaces are normal at a point if their gradients are normal there, i.e. if the dot product of their gradients is zero.