MATH 233 LECTURE 19 (§14.7): MAXIMA AND MINIMA OF MULTIVARIABLE FUNCTIONS

This lecture gives a bit of extra explanation behind the algorithm we will be using to find maxima and minima. You are only responsible for what is in the book.

- Given a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can consider $\vec{v} = \langle v_x, v_y \rangle$ as a 1-by-2 matrix and matrix-multiply: $\vec{v}M = \langle v_x, v_y \rangle \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \langle av_x + cv_y, bv_x + dv_y \rangle$.
- At a local maximum or minimum of a *differentiable* function $f(x, y)$, the slope must be zero in all directions – in particular, the partials f_x and f_y are zero. This is called a *stationary point*. (The only other kind of point where you can have a local extremum is a *singular point*, i.e. a point where f is *not* differentiable.)
- Now suppose you know $f_x(0, 0) = 0 = f_y(0, 0)$. Is $(0, 0)$ a local maximum? minimum? As in the 1-variable setting, we need to look at second (partial) derivatives to decide. Unlike the 1-variable case, there is a 3rd possibillity: that $(0, 0)$ is a *saddle point* (maximum in one direction and minimum in the other).
- Let $\hat{u} = \langle u_x, u_y \rangle = \langle \cos\theta, \sin\theta \rangle$ be a unit vector, $f(x, y)$ a function of two variables, and $M_f(x, y) = \begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix}$. (This matrix depends on x and y, and is "symmetric" since $f_{xy} = f_{yx}$. Write M_f for $M_f(0, 0)$.) By a straightforward computation which we will do in class, the second directional derivative $D^2_{\hat{u}}f = \hat{u}M_f(x, y) \cdot \hat{u}$. In particular, $(D^2_{\hat{u}}f)(0, 0) = \hat{u}M_f \cdot \hat{u}$ gives the concavity of f at $(0, 0)$ in the $\hat{u}$-direction.
- We would like to find in what directions θ the concavity is maximized and minimized at $(0, 0)$. So set $0 = \frac{d}{d\theta}(\hat{u}M_f \cdot \hat{u}) = \frac{d\hat{u}}{d\theta}M_f \cdot \hat{u} + \hat{u}M_f \cdot \frac{d\hat{u}}{d\theta} = 2\hat{u}M_f \cdot \frac{d\hat{u}}{d\theta}$ (here I am using $\vec{v}M \cdot \vec{w} = \vec{w}M \cdot \vec{v}$ for a symmetric matrix M), which gives

$\hat{u}M_f \perp \frac{d\hat{u}}{d\theta}$. But since $\frac{d\hat{u}}{d\theta} \perp \hat{u}$, this means that $\hat{u}M_f$ is parallel to $\hat{u}$, i.e.

$$\hat{u}M_f = \alpha\hat{u}$$

for some $\alpha \in \mathbb{R}$. (Note that then $(D^2_{\hat{u}}f)(0,0) = \alpha\hat{u} \cdot \hat{u} = \alpha$ is the concavity in the direction $\hat{u}$.)

- That is, the concavity $(D^2_{\hat{u}}f)(0,0)$ is greatest/least when $\hat{u}$ is an "eigenvector" or "stretch vector" of M_f. Here α is the "eigenvalue" or "stretch coefficient" of M_f. A symmetric 2×2 matrix has two of these – say, α_1 and α_2 – and the determinant

$$\det(M_f) = \alpha_1\alpha_2$$

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- Another way to say this is that $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2$ (i.e. $\det(M_f)$) is equal to the product of the maximum and minimum concavities of f at $(0,0)$. So *if the determinant is negative, then these concavities are positive and negative, respectively, and we have a saddle point.*
- *If the determinant is positive, then either both concavities are positive (local minimum) or both are negative (local maximum).* You can tell which one by looking at the concavity in any direction – say $f_{xx}(0,0)$ – since this is between α_1 and α_2 hence shares their common sign.
- What if the determinant is zero? Then our 2-variable second derivative test is *inconclusive.*