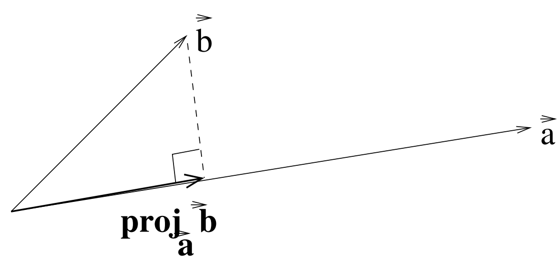
MATH 233 LECTURE 2 (§§12.3-4): DOT AND CROSS PRODUCTS

Dot product.

- If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then $\vec{a} \cdot \vec{b} := a_1 b_1 + a_2 b_2 + a_3 b_3$. This is a number (scalar). Notice that $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$, and that dot product satisfies commutative and distributive laws.
- Geometric interpretation: $\vec{a} \cdot \vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. (In class, I explained one way to check this.) In particular, $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$. More generally, this formula gives you a way to find θ.
- "Direction angles" are the angles α, β, γ that $\vec{a}$ makes with the standard basis vectors $\hat{\imath} = \langle 1, 0, 0 \rangle$, $\hat{\jmath} = \langle 0, 1, 0 \rangle$, $\hat{k} = \langle 0, 0, 1 \rangle$. The "direction cosines" are (you guessed it) $\cos\alpha$, $\cos\beta$, $\cos\gamma$. The sum of their squares is 1 (why?).
- The "vector projection of $\vec{b}$ onto $\vec{a}$" is the vector $\mathbf{proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\vec{a}$, and its length $\mathrm{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$ is called the "scalar projection" or "component of $\vec{b}$ along $\vec{a}$":



- In physics, work is computed by taking the dot product $W = \vec{F} \cdot \vec{D}$ of force and distance, assuming the force is constant over that distance. (Distance here is a vector pointing from start to finish.)

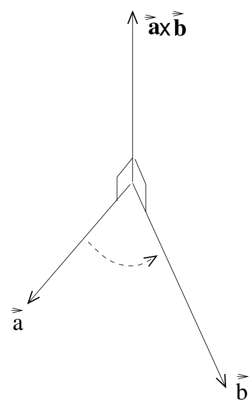
Cross-product.

- In the same notation as above (for $\vec{a}$ and $\vec{b}$),

$$\vec{a} \times \vec{b} := \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

  Unlike the dot product, this is a vector.

- Since $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$, in the direction according to the right-hand rule:



- The length

$$\left\|\vec{a} \times \vec{b}\right\| = \|\vec{a}\| \left\|\vec{b}\right\| |\sin\theta| \tag{1}$$

  depends on the angle between $\vec{a}$ and $\vec{b}$. This is biggest when they are perpendicular, and zero when they are parallel.

Quick proof of equation (1)**.** This is better than Stewart's if you like summation notation. (If you don't, read Stewart's, or skip it.) In the sums, i and j run from 1 to 3.

$$
\begin{aligned}
\left\|\vec{a}\times\vec{b}\right\|^2 + (\vec{a}\cdot\vec{b})^2 &= (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) + \left(\sum_i a_i b_i\right)^2 \\
&= \sum_{i<j}(a_i b_j - a_j b_i)^2 + \left(\sum_i a_i^2 b_i^2 + \sum_{i\neq j} a_i b_i a_j b_j\right) \\
&= \sum_{i<j} a_i^2 b_j^2 + \sum_{i<j} a_j^2 b_i^2 - 2\sum_{i<j} a_i b_j a_j b_i + \left(\sum_i a_i^2 b_i^2 + 2\sum_{i<j} a_i b_i a_j b_j\right) \\
&= \left(\sum_i a_i^2\right)\left(\sum_j b_j^2\right) \\
&= \|\vec{a}\|^2 \left\|\vec{b}\right\|^2 .
\end{aligned}
$$

Since $\vec{a}\cdot\vec{b} = \|\vec{a}\| \left\|\vec{b}\right\| \cos\theta$, we have

$$
\left\|\vec{a}\times\vec{b}\right\|^2 = \|\vec{a}\|^2 \left\|\vec{b}\right\|^2 (1-\cos^2\theta) = \|\vec{a}\|^2 \left\|\vec{b}\right\|^2 \sin^2\theta,
$$

and taking square roots on both sides gives (1).