MATH 233 LECTURE 24: DOUBLE INTEGRALS (A FIRST LOOK)

- Recall the single-variable integral: to integrate a function $f : [a,b] \to \mathbb{R}$, subdivide $[a,b] = \cup_{i=1}^{m}[x_{i-1}, x_i]$ where $x_0 = a, \ldots, x_i = a + \frac{b-a}{m}, \ldots, x_n = b$. (Denote $\frac{b-a}{m}$ by Δx.) Then take any point $x_i^* \in [x_{i-1}, x_i]$ in each interval, and write down the *Riemann sum* $\sum_{i=1}^{m} f(x_i^*)\Delta x$. The limit of this sum as $m \to \infty$ is written $\int_a^b f(x)dx$ and called the *definite integral* of $f(x)$ on $[a,b]$. Provided f is bounded with only finitely many discontinuities, this limit exists and is independent of the choice of the x_i^*. Geometrically, it gives the area over the x-axis and under the graph of $f(x)$ (where f is positive), minus the area under the x-axis and over the graph of $f(x)$ (where f is negative). If $f = F'$, then one has the Fundamental Theorem of Calculus: $\int_a^b f(x)dx = F(b) - F(a)$, since you are adding up $F'(x_i^*)\Delta x$ (little changes in F).
- Double integrals over rectangles: let $R = [a,b] \times [c,d]$ be a rectangle, $f : R \to \mathbb{R}$ a function. Subdivide R into subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ where (with $\Delta x = \frac{b-a}{m}$, $\Delta y = \frac{d-c}{n}$) $x_i = a + i(\Delta x)$, $y_j = c + j(\Delta y)$. Pick a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$ in each rectangle, and add up the volumes of the skinny boxes with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. This gives the *double Riemann sum*

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A,$$

  where $\Delta A = \frac{(b-a)(d-c)}{mn}$.
- Notice that different kinds of Riemann sums are possible: for example, we could take (x_{ij}^*, y_{ij}^*) to be the midpoint of R_{ij}, or its upper-right corner, etc. We could also take it to be the point in R_{ij} where f is largest [resp. smallest], which gives the so-called *upper* and *lower* Riemann sums.

- We say that f is *integrable* (on R) if the limit

$$\iint_R f(x,y)dA := \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

  exists (and is independent of the choices of the points). In that case this is called the *double integral* of f over R.
- If f is bounded on R and continuous there (except on finitely many smooth curves), then f is integrable on R.
- $\iint_R f(x,y)dA$ gives the volume under the graph of $z = f(x,y)$ and over the xy-plane, minus the volume under the xy-plane and over the graph. (If f is positive, you may thus view $\iint f(x,y)dA$ as calculating the volume of the solid described by $0 \le z \le f(x,y)$, $a \le x \le b$, $c \le y \le d$.) By this logic, you can deduce that the upper Riemann sum is bigger than $\iint_R f(x,y)dA$, and the lower Riemann sum always smaller.
- Linearity property: $\iint_R (af(x,y)+bg(x,y))dA = a\iint_R f(x,y)dA + b\iint_R g(x,y)dA$
- Comparison property: If $f \le g$ on R, then $\iint_R f(x,y)dA \le \iint_R g(x,y)dA$. One consequence is that if $m \le f(x,y) \le M$ on R, then $mA(R) \le \iint_R f(x,y)dA \le MA(R)$ (where $A(R)$ is the area of R).
- Additivity on rectangles: if $R = R_1 \cup R_2$, and R_1 and R_2 only meet along a segment, then $\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$.
- Watch out for functions which go to ∞ along some subset of R. These are (obviously) not bounded and so are not integrable by the above definition. (Though you can successfully integrate them sometimes by taking a limit of integrals over subsets of R that omit the bad subset.)