MATH 233 LECTURE 27: NONRECTANGULAR DOUBLE INTEGRALS

- Let D be a closed, bounded region in the xy-plane, with piecewise smooth boundary. Take $R = [a, b] \times [c, d]$ to be a rectangle containing D. If $f(x, y)$ is a function on D, construct a function on R by

$$F(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D \end{cases}.$$

The integral of f over D is then defined to be the integral of F over R.

- How do we evaluate such integrals? It depends on D. For instance, we shall say that D is y-simple if each line parallel to the y-axis intersects D in a single interval (or a point, or not at all) – that is, if there are functions ϕ_1, ϕ_2 on $[a, b]$ such that $D = \{(x, y) \,|\, a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x)\}$ is the region sandwiched between their graphs. In this case,

$$\iint_D f(x,y)dA = \iint_R F(x,y)dA = \int_a^b \left(\int_c^d F(x,y)dy \right) dx$$
$$= \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy \right) dx.$$

- Similarly, if D is x-simple, i.e. $D = \{(x, y) \,|\, \psi_1(y) \le x \le \psi_2(y)\}$, then

$$\iint_D f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy.$$

- Some regions are both x-simple and y-simple, and so you have to judge which is the easier way to perform the iteration, just as in the rectangular case. Some regions are neither x- nor y-simple, but can be cut up into such regions, and the integrals summed at the end.

- Beware of switching the order of integration in these no-rectangular cases: this requires drawing the region. This will be discussed more in the next class.