MATH 233 LECTURE 28: DOUBLE INTEGRALS AND VOLUME

- Recall the definition of the double integral of a function f on a closed bounded region D: one simply fits D into a rectangle R, extends f to a function F on R which is zero outside D, and sets

$$\iint_D f(x,y)dA := \iint_R F(x,y)dA.$$

  We defined x-simple and y-simple regions D and explained how to use this to translate this definition into an iterated integral, e.g. $\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dy\,dx$. (See Lecture 27.)
- The main point of this lecture is simply that (for $f \geq 0$) this double integral computes the volume under the graph of $z = f(x,y)$ over D. (This implies also that $\iint_D 1\,dA$ computes the area of D.) There are various other interpretations: for example, to compute the average value of f over D, you compute $\iint_D f\,dA$ and divide by $\iint_D 1\,dA$.
- For regions that are more natural in polar coordinates (r,θ) there is a more convenient way to compute these double integrals. Instead of thinking of a "little bit of area" dA as $dx \cdot dy$ (that of a little rectangle), you use an infinitesimal polar rectangle with area $r \cdot dr \cdot d\theta$. (One side has length $rd\theta$, the other dr.) Rewriting f as a function of (r,θ) by $f(x,y) = f(r\cos\theta, r\sin\theta)$, this now becomes an iterated integral in r and θ.
- The simplest case is where the region D is itself a (finite, not infinitesimal) polar rectangle: $D = \{(r,\theta) \,|\, a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$. (For exmple, a disk is a "polar rectangle" in this sense, with $\alpha = 0$, $\beta = 2\pi$, $a = 0$, $b =$ the radius of the disk.)

The integral then becomes

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_a^b F(r, \theta)rdrd\theta,$$

where $F(r, \theta) := f(r\cos\theta, r\sin\theta)$. To be continued in the next lecture . . .