MATH 233 LECTURE 29: MORE POLAR INTEGRATION

- Recall that to integrate a function $f(x, y)$ over a polar rectangle $P = \{(r, \theta) \mid a \leq r \leq b,\ \alpha \leq \theta \leq \beta\}$, we can write

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_a^b F(r,\theta) r dr d\theta,$$

where $F(r, \theta) := f(r\cos\theta, r\sin\theta)$.

- We can also accomodate more general regions D which are "r-simple" (given by $\alpha \leq \theta \leq \beta$, $\phi_1(\theta) \leq r \leq \phi_2(\theta)$) or "$\theta$-simple" ($a \leq r \leq b$, $\psi_1(r) \leq \theta \leq \psi_2(r)$) via the iterated integrals

$$\int_\alpha^\beta \int_{\phi_1(r)}^{\phi_2(r)} F(r,\theta) r\, dr\, d\theta \quad \text{resp.} \quad \int_a^b \int_{\psi_1(r)}^{\psi_2(r)} F(r,\theta) d\theta\, r dr.$$

The remainder of the lecture will consist of examples, except for:

- An important application (for probability): area under the bell curve $y = e^{-x^2/2}$. Set $I = \int_0^\infty e^{-x^2} dx := \lim_{b\to\infty} \int_0^b e^{-x^2} dx$, and let V be the volume under the surface $z = e^{-x^2-y^2}$. On the one hand, we will see that $V = 4I^2$. On the other, using polar integration, one easily shows that $V = \pi$. Conclude that $I = \frac{\sqrt{\pi}}{2}$, and so the area under the original bell curve is $\sqrt{2\pi}$.