MATH 233 LECTURE 30: APPLICATIONS OF DOUBLE INTEGRALS

- Mass: consider a lamina (flat sheet) covering a region D in the xy-plane, with mass density function $\rho : D \to \mathbb{R}$. The total mass m of the lamina is given by $\iint_D \rho(x, y)\, dA$.
- Center of mass: define "moments" of the lamina about the y- and x-axes by $M_y = \iint_D x\rho(x, y)\, dA$, $M_x = \iint_D y\rho(x, y)\, dA$. The center of mass is then given by $(\bar{x}, \bar{y}) := (M_y/m,\, M_x/m)$.
- Moment of inertia: the kinetic energy of a particle of mass m traveling on a circle of radius r with angular velocity ω is given by $\frac{1}{2}mr^2\omega^2$. The portion $I := mr^2$ is called the moment of inertia. By integrating, we can define moments of inertia of a lamina about the y-axis, x-axis, and origin: $I_y = \iint_D x^2\rho(x, y)\, dA$, $I_x = \iint_D y^2\rho(x, y)\, dA$, $I_0 = \iint_D (x^2 + y^2)\rho(x, y)\, dA$. These measure how hard it is to change the angular velocity of the lamina about the y-axis, x-axis, and origin.

Probability.

- Let X be a random variable with probability density (distribution) function $f : \mathbb{R} \to \mathbb{R}$, then the probability that X lies in the interval $[a, b]$ is given by $P(a \leq X \leq b) = \int_a^b f(x)dx$. (Of course, the integral over all of $\mathbb{R}$ must give 1, or 100%).
- Expected value: $\bar{X} := \int_{-\infty}^{\infty} xf(x)dx$.
- More generally, if X and Y are random variables with joint probablity density function $f : \mathbb{R}^2 \to \mathbb{R}$, then the probablity that X and Y lie in some region D is just $P((X, Y) \in D) = \iint_D f(x, y)dA$. For example, if this region is $D =$

$\{(x, y) \,|\, x \leq y\}$, then this integral computes the probability that X is smaller than (or equal to) Y.

- Expected values: $\bar{X} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy$, $\bar{Y} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) \, dx \, dy$.
- Independence: if the joint probability density function is a product, $f(x, y) = F(x)G(y)$, then the two variables are independent: that is, the probability of X being in some range is independent of the value of Y.